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THE FINITE ANALYTIC METHOD

by

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and

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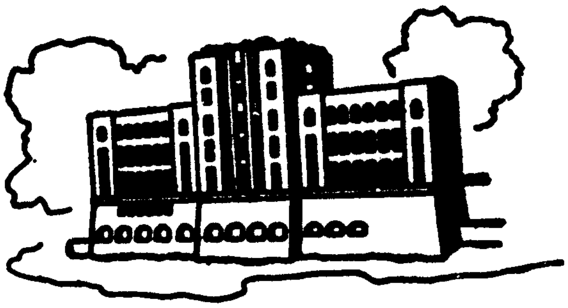

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VOLUME 3

THE FINITE ANALYTIC METHOD

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THE FINITE ANALYTIC METHOD

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PREFACE

The Finite Analytic Method

This monograph contains the fundamental development of the new numerical method called the "Finite Analytic" method. The finite analytic method differs from the finite difference method and the finite element method. The basic idea of the finite analytic method is the incorporation of local analytic solutions in the numerical solution of linear or nonlinear partial differential equations. In the finite analytic method, the total problem is subdivided into a number of small elements. The local analytic solution is obtained for the small element in which the governing equation, if nonlinear, is linearized. The local analytic solutions are then expressed in algebraic form and are overlapped to cover the entire region of the problem. The assembly of these local analytic solutions, which still preserves the overall nonlinearity of the governing equation, results in a system of linear algebraic equations. The system of algebraic equations is then solved to provide the numerical solutions of the total problem.

Unlike the finite difference method, the finite analytic method does not tamper with the differentials or the derivatives of the governing equation, nor does the analytic method need the shape function which is made to satisfy the integral form of the governing equation, as in the finite element method. The finite analytic solution obtained from the finite analytic method is differentiable. As a result, the derivative of the solution obtained analytically is much more reliable. In this monograph the finite analytic solution is shown to be stable, even when the highest derivative term of the partial differential equation is multiplied by a small factor, such as one over Reynolds number. It is also shown that the finite analytic solution for Navier-Stokes equations at high Reynolds numbers automatically provides a gradual shift of the upwinding effect. Therefore the finite analytic solution accurately simulates the effect of convection and eliminates the false numerical diffusion that would occur in the upwinding difference or unidirection difference used in the finite difference or the finite element methods. The computational time for the finite anayltic solution is shown to be about equal to that of the finite difference method. In certain cases, due to the stability of the system of algebraic equations derived in the finite analyitc method, the overall computational time can be even less. The finite analytic solution derived in the present analytic mehtod is in its most elementary form in terms of accuracy. But it has already been shown to be sufficient for the problems under consideration. Further accurate finite analytic formulae can be derived and are indicated in the monograph.

The finite anlytic method was developed in early 1977, when Dr. Peter Li was then a graduate student working on his doctoral dissertation with me. He had been having difficulty in obtaining convergence of a system of finite difference algebraic equations derived from the Navier-Stokes equations for two-dimensional turbulent flow with a second-order turbulent model. I conceived the finite analytic method one night and solved the simple two-dimensional Laplace equation. Li then carried the finite analytic method to the unsteady diffusion equation and nonlinear ordinary differential equations and complted his Ph.D. dissertation in 1978. In 1981, Messrs. Mohamad Zahed Sheikholeslami, Bahram Khalighi, and Kanwerdip Singh developed the finite analytic method further by solving the ordinary and partial differential equations and the Navier-Stokes equations with primitive variables. This bound volume essentially contains the research results of Messrs. Sheikholeslami, Khalighi, Singh, and myself.

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ACKNOWLEDGEMENT

I would like to acknowledge Messrs. Mohamad Zahed Sheikholeslami, Bahram Khalighi, and Kanwerdip Singh for taking the finite analytic method as their theses, respectively. Without them, the finite analytic method could not have been developed and understood as it is today.

I would also like to thank my colleagues, Professors V. C. Patel, David C. Chou, T. F. Smith, Allen Chwang, and K. Atkinson, for their encouragement and criticisms of the method. I would also like to thank Drs. William D. Mcnally, Peter M. Sockol, Gary Johnson, and J. J. Adamczyk of NASA Lewis Center for taking a keen interest in and supporting the continuation of the development of the finite analytic method. My thanks also go to Dr. Melvyn Ciment, of the Applied Mathematical Division, and Dr. Ronald W. Davis, of the Fluid Engineering Division, of the National Bureau of Standards, for their in-deep discussion of the method and to Drs. Oscar P. Manley of the U. S. Department of Energy and P.C. Lu of the University of Nebraska for their discussion and encouragement. This work is, in part, supported by NASA Grant No. N.S.G. 3305 and U. S. Department of Energy Grant No. DE-AC02-79ER-10515.A000. The support of the University of Iowa Computer Center and Division of Energy Engineering, The University of Iowa, is also acknowledged.

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PART I

Application of Finite Analytic Method to The Numerical Solution of Two-Point Boundary Value Problems of Ordinary Differential Equations

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LIST OF SYMBOLS

$A(y',y,x)$	Nonlinear term of the ordinary differential equation (See Equation (IV-1))
A	Average value of the $A(y',y,x)$ in each small subregion. (See Equation (IV-3))
$B(y',y,x)$	Nonlinear term of the ordinary differential equation (See Equation (IV-1))
B	Average value of $B(y',y,x)$ in each small subregion. (See Equation (IV-3))
$B_a(\ )$	Boundary condition at point $x = a$
$C(y',y,x)$	Nonhomogeneous part of the nonlinear ordinary differential equation (See Equation (IV-1))
C	Average value of $C(y',y,x)$ in each small subregion (See Equation (IV-3))
C_1, C_2	Constants of integration
C_{i-1}, C_{i+1}, C'_{i-1}, C'_{i+1}	Coefficients of the FA formula in terms of the nodal points in the total region (See Equation (III-6))
C_S, C_N, C'_S, C'_N	Coefficients of FA formula for a typical line segment
D_{i-1}, D_{i+1}, D'_{i-1}, D'_{i+1}	Coefficients of the FA formula for derivatives in terms of the nodal points in the total region (See Equation (III-6))
D_S, D_N, D'_S, D'_N	Coefficients of the FA formula for derivatives in terms of the nodal points in the total region (See Equation (III-6))
f	Dependent variable of the Falkner-Skan equation (Chapter VI)

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f', f'', f'''	First, second, and third derivatives of the dependent variable f of the Falkner-Skan equation
G	Nonhomogeneous term of the ordinary differential equation
g	Dependent variable of the linearized Falkner-Skan equation substituted for f'
h	Grid size
L( )	Ordinary differential operator
m, m_1, m_2	Roots of the characteristic equation of the linear ordinary differential equation (See Equation (IV-5))
p, q	real and imaginary parts of the roots of the characteristic equation (See Equation (IV-6-1)
x	Independent variable
y, y', y''	Dependent variable of the ordinary differential equation and its derivatives
y_S, y_P, y_N	Functional values of the south, middle, and north points of each finite sub-region respectively
y_{i-1}, y_i, y_{i+1}	Value of dependent variable for nodal points in the total region
α_1, α_2	Lower and upper bounds of the independent variable
β	Pressure gradient parameter in the Falkner-Skan equation (Chapter VI)
γ	Relaxation factor used in Chapter IV

CHAPTER I

INTRODUCTION

Two-point boundary value problems associated with systems of linear and nonlinear ordinary differential equations occur in many branches of mathematics, engineering, and the various sciences. In these problems boundary conditions are specified at the end points of the problem interval, and a solution of the differential equations over the interval is sought which satisfies the given boundary conditions.

Generally, for boundary value problems, if the differential equation is nonlinear or it is linear with variable coefficients, the construction of a solution, even though it may be known to exist and to be unique, is difficult, and the integration of the differential equation must often resort to a numerical approach.

Several numerical methods have been developed for solving ordinary differential equations of boundary value problem, which may be divided into two main approaches, discretization methods and integral methods. The discretization methods are based on discretizing the problem domain into small regions. Depending on how the approximate solution is devised in the small subregions, there are several discretization methods. For example, finite difference (FD) methods (1), (2), finite element (FE) method (3), and the recently developed finite analytic (FA) method (4).

The integral methods are based on approximating the solution over the whole interval by a series. Each term of the series is usually a polynomial or a suitable function that satisfies the boundary conditions. The coefficients of the series are determined by substituting the series into the differential equation, and minimizing the residual (5).

In the FD method, the total region is broken up into finite subregions by a finite number of discrete points. The finite difference is obtained from a truncated Taylor series expansion to provide approximately the relation between the dependent variable and its derivative at a chosen point, and its neighboring points. The differential equation at each point is approximated by a difference equation. Therefore, for n discrete points, n algebraic equations are obtained, relating the unknown dependent variables with its ngighboring points. This system is readily solved if the algebraic system is linear. If it is nonlinear, the equation is linearized and the solution is obtained with a suitable iterative method (6).

In the FE method, the first step is to subdivide the problem domain into small subregions. Then an approximate functional form connecting the unknown nodal values of the dependent variables in the subregion is chosen to represent the solution in each subregion. These approximate functions (shape functions) normally are polynomials because of their simplicity. The approximate function is then made to satisfy the governing equation in an integral form in each subregion. The most commonly used forms of the integrals are the weighted residuals integral (WR) and the variational form of the governing equation (7). The weighted residuals integral is based on minimization of the residual in the subregion when the approximate solution is substituted into the integral of the differential equation governing the problem. Other schemes are possible to achieve the aim of minimization of resuduals such as collocation, sub-domain, least squares, and Galerkin methods (8). Minimizing the residual leads to an algebraic equation describing the behavior of an element. For all the elements a set of linear (or nonlinear) simultaneous algebraic equations are obtained relating the value at each nodal point with its neighboring points. The set of algebraic equations is solved as in the case of the FD method.

The recently developed finite analytic (FA) method is neither a finite difference nor a finite element method.

The FA method utilizes local analytic solutions of the differential equation obtained for small regions regions that form the total region considered in the problem. The FA numerical solution of the problem is then made of all the local analytic solutions. If the differential equation is nonlinear or linear with complex variable coefficients, the FA method divides the problem into many subintervals. In each subregion, the nonlinear terms are locally linearized, and the complex variable coefficients are replaced by a local constant. By solving the differential equation in each subregion, a relation between the unknown dependent variable at nodal points in the subregion is obtained. By repeating this procedure for each subregion, a system of algebraic equations is obtained relating the unknown dependent variable at each point with values of surrounding points. The system of the algebraic equations is then solved as in the FD or FE methods.

The methods described above have been used extensively in numerical solutions of differential equations. The FD method is easy to handle, but, due to the approximation made for the derivatives, the method may not provide accurate solutions, and sometimes the system of algebraic equation derived from a particular finite difference scheme is unstable. The FE method is relatively stable and can treat very complex boundary conditions, but it needs a considerable more amount of mathematics than the finite difference (FD) method. Also, it has difficulty in treating the boundary conditions specified at infinity.

The aim of this study is to extend the FA method to boundary value problems of second order ordinary differential equations, and to examine the convergence, stability, and accuracy of the FA method. A comparison of the finite analytic solution with solutions obtained from the FD method is given for several numerical examples.

CHAPTER II

LITERATURE REVIEW

Although there are many works which have studied numerical solution of two point boundary value problems, there are few works that bear resemblance to the FA method (4), (9), (10), which shall be investigated in the present study. Before reviewing details of previous works, different methods used in solving both linear and non-linear two point boundary value problems will be briefly described. It should be mentioned that most methods used to solve nonlinear boundary value problems invoke local linearization at some stage of the numerical calculation.

II-1. Numerical Methods

II-1-1. The Method of Weighted Residuals

The method of weighted residuals, sometimes known as the method of undetermined coefficients, is essentially an integral method of obtaining solutions to differential equations. In this method, the unknown solution is expanded in a set of trial functions with adjustable constants, which are chosen to give the best solution. The trial functions are a family of functions satisfying the boundary conditions of the original problem. The substitution of these trial functions and their derivatives into the original equation gives a residual equation describing the error in the solution interval. If the trial function were the exace solution, the residual would be zero. The constants in the trial function are chosen in such a way that the residual is forced to be zero in an average sense.

There are several ways for computing the coefficients of the trial series; for example, the collocation method, the least squares method, and the Galerkin method. In the collocation method (11), (12), the coefficients are determined by the requirement that the trial function has to satisfy exactly the governing equation at chosen locations. The numerr of locations chosen should be equal to the number of unknown coefficients. In the least squares method, the weighting function is chosen to be the residual. Thus, the method is based on choosing coefficients of trial function such that the integral of the square of the residual over the interval under consideration can be minimized.

One of the best known approximate methods was developed by Galerkin in 1915 (13). In this method, the weighted functions are chosen to be the trial functions. The trial functions must be chosen as members of a complete set of orthogonal functions. A set of orthogonal functions is complete if any function of a given class can be expanded in terms of the set. Thus, the Galerkin method forces the residual to be zero by making it orthogonal to each member of a complete set of functions.

II-1-2. Finite Difference Method

Among different methods suggested for solving boundary value problems, the FD methods are more frequently used (1), (2). The FD method mentioned in Chapter I is based on the difference approximation of derivatives derived from truncated Taylor series expansions, thus converting the ordinary differential equation into a set of algebraic equations, thus provides the numerical calculation of the ordinary differential equation.

Although the FD method is not the present FA method, there are several studies combining the FD methods with the analytic solutions of problems (9), (10), which have some resemblance to the FA method.

Allen and Southwell (10), in seeking a numerical solution for the two-dimensional motion of a viscous fluid past a fixed cylinder, solved a nonlinear partial differential equation in terms of stream function and vorticity. This equation is linear, and is solved by a finite difference method yielding a relationship between the stream function at a point and its neighboring nodes. The vorticity equation is solved by a "two diagram technique" in which the stream function and the vorticity are modified alternately. That is, the linearized vorticity equation is divided into two parts, each of which is an ordinary differential equation because it contains only terms with derivatives in one direction. The analytic solution is then obtained for each ordinary differential equation. These analytic solutions are used to modify the finite difference approximation of the vorticity equation. The modified finite difference equations include the exponential terms that are obtained from the analytic part of the solution.

Recently, Dennis and Hudson (9) exploited this idea further to obtain a higher order approximation to second order partial differential equations. Again, the partial differential equation is divided into two parts, each part being an ordinary differential equation. These equations are solved in two normal directions. The two analytic solutions are then matched at the point of intersection of the two normal lines. This process leads to the finite difference approximation to the original problem. The above methods are similar to the FA method in the sense that both invoke the analytic solutions. However, in the present FA method, the finite difference approximation is not used. The FA numerical solutions are obtained from the assembly of all local analytic solutions.

II-1-3. Finite Element Method

In the FE method (7), which is discussed in Chapter I, the differential equation is written in its variational form, known as a functional or an integral function, is to be minimized in each finite element. Therefore, after discretizing the whole region into small finite sub-regions, the solution will be represented by an approximate function (shape function) with unknown coefficients. Substitution of this approximate function into the integral function and minimizing it yields a system of algebraic equations from which the unknown coefficients can be obtained. This system can be solved as in the case of the FD method.

II-1-4. Finite Analytic Method

Direct utilization of the local analytic solution of the linearized problem in the numerical solution of the ordinary differential equations has not been used in the above methods. The idea of incorporation of local analytic solutions of the linearized equation in the numerical solution of boundary value problems, which is the basic principle or the FA method first introduced by Chen and Li (4). Although most of their work was devoted to the treatment of partial differential equations, there is a short discussion about ordinary differential equations. The example considered was the Falkner-Skan problem (14): the solution of $f''' + ff'' + \beta(1-f'^2) = 0$, a nonlinear differential equation with boundary conditions $f'(0) = f(0) = 0$ and $f'(\infty) = 1$. The governing equation is linearized and integrated locally.

However, the integrand is approximated by a second degree polynomial. The problem is then cast into an initial value problem, and solved by a shooting technique. In the shooting method, only f and f' are given at the boundary $\eta = 0$. Therefore, $f''(0)$ will have to be determined wuch that the solution satisfies the far field boundary condition of $f'(\infty) = 1$ at $\eta = \infty$. This shooting algorithm is similar to that used by Carnahan et.al (6). However, instead of the Runge-Kutta integration scheme, the FA formulation was used.

In the present investigation, the shooting technique is not used. All problems are treated as boundary value problems. In solving the Falkner-Skan problem, the linearized equation in f' is solved as a boundary value problem with the known boundary conditions, i.e., $f'(0) = 0$, $f'(\infty) = 1$ when the far field boundary condition $f'(\infty) = 1$ in the calculation is replaced by a finite domain. That is, it is assumed that $f'(\eta_\infty) = 1.0$. This Falkner-Skan problem is solved in detail and discussed in Chapter VI.

II-2. Methods of Solving Boundary Value Problems

So far different numerical schemes for solving boundary value problems have been considered. In solving boundary value problems with higher order finite difference schemes, such as Runge-Kutta, the problem is usually cast into a series of first order initial value problems. To solve these equations, the initial condition for each equation is needed. But since the original conditions are specified at the boundaries, some of the initial conditions will be missing. There are different methods of obtaining these missing initial conditions. The most widely used technique of finding a missing initial condition is the shooting method, which will be discussed briefly.

II-2-1. Shooting Method

The methods of finding the missing initial conditions can be systematically applied. One of the most useful methods is the method of adjoints for the linear equation. This method is based on associating with every set of linear ordinary differential equations a companion set of equations called the adjoint equations. The adjoint equations are defined as the set of homogeneous linear ordinary differential equations whose matrix of coefficients is the negative transpose of the matrix of the original set of linear ordinary differential equations. The initial and terminal boundary conditions of adjoint equations are related to the initial and terminal boundary conditions of the original system by a certain identity. By solving the adjoint equations, the missing initial conditions are found directly. For nonlinear two point boundary value problems, the method of adjoint equations can also be used iteratively after the nonlinear term is locally linearized. The method for nonlinear problems does not compute the missing initial conditions, but rather computes corrections to the trial values for the missing initial conditions. For a complete discussion of this method and similar methods, refer to (15).

II-2-2. Invariant Imbedding

Invariant imbedding is another technique that can be exploited to find the missing initial conditions (16),(17). Consider a differential equation which is to be solved in the domain $(0,t_f)$. Instead of only considering a single problem with an interval of $(0,t_f)$, the invariant imbedding approach is to consider a family of problems that consist of a variable interval (0,a) where a ranges from zero to the value of t_f. Then the problems are solved first for a small interval of (0,<u>a</u>), where a is close to zero. Since the differential equation had almost zero interval, the missing initial condition may be obtained by a Taylor series. Expressed in this way the original two point boundary value problem becomes an initial value problem in the invariant imbedding formulation. The family of problems is then formed by increments of the interval length. This is the essence of the method of invariant imbedding.

II-3. Linearization Technique

Many boundary value problems occuring in science and engineering are nonlinear. Therefore, to be solved by the FD or FA method, it is necessary to linearize them. There are different methods to overcome this difficulty. One way is the interval averaging approximation; i.e., the nonlinear terms are replaced by an integral average of their values over each small subregion. Obviously, to start the linearization, an initial guess for the nonlinear terms is needed, which makes the process an iterative one. Quasilinearization is a more standard way of linearizing the nonlinear forms. In the quasilinearization technique, instead of being solved directly, the nonlinear differential equation is solved recursively with an approximated linear differential equation. To illustrate the quasilinearizations, consider the nonlinear second order differential equation $y''(x) = f(y(x), y'(x))$. Here $f(y,y')$ denotes the function which contains nonlinear terms. The quasilinearization process starts with expanding f in Taylor series in terms of the functions y and y' around a given function of $y_0 = y_0(x)$ with second and higher order terms of the series expansion omitted. Here $y_0(x)$ is a chosen function which satisfies the boundary conditions and is used as the initial guess of the solution. Replacing f by its Taylor series expansion gives a linear differential equation with variable coefficients. Solving this equation with the initial approximated function $y_0(x)$, a better approximation to the solution will be obtained, say $y_1(x)$. Replacing y_0 by y_1 and repeating this procedure, further improved solution will be obtained. This iterative procedure is very similar to the method of successive substitution or the Newton-Raphson method, but instead of roots of an algebraic equation, it contains the solution of a differential equation. This method was originally developed by Bellman and Kalaba (18) and has been used for solving nonlinear two point boundary value problems by many authors (19), (20). In Chapter V and Chapter VI of the present study, comparison is made between quasilinearization and integral averaging approximations.

CHAPTER III

PRINCIPLES OF FINITE ANALYTIC METHOD

The basic idea of the finite analytic method is the incorporation of analytic solutions in the numerical solution of differential equations. To illustrate the basic principles of the FA method for solving boundary value problems of ordinary differential equations, consider a second order ordinary differential equation:

$$L(y(x)) = G \qquad a < x < b$$

subject to boundary conditions

$$B_a(y_a, y_a') = 0 \qquad B_b(y_b, y_b') = 0 \qquad \text{(III-1)}$$

over an interval [a,b] as shown in Figure (III-1-a). L may be a linear or nonlinear second order differential operator, G is the nonhomogeneous term of the ordinary differential equation. The boundary conditions are specified at $x = a$ and $x = b$.

The objective of the FA method is to obtain a numerical solution for such a boundary value problem, when the analytic solution of the problem is difficult to obtain, due to the nonlinearity of the differential equation or the complexity of the coefficients.

The first step in applying the FA method is to subdivide the total region (line x) of the problem into n finite subregions with finite line elements of length h, y_i, denoting the nodal value of the dependent variables at i^{th} node where $i = 1,2,\ldots\; n + 1$.

Consider a line element of length 2h (Figure III-1-b). In this small line element, if the differential equation is linear with complicated coefficients the coefficients are made constant locally, and if the differential equation is nonlinear, the nonlinear terms are linearized and variable coefficients made constant locally. The local constant used in linearization varies from interval to interval. The analytic solution for the locally linearized problem can be obtained easily. If the line elements are small, the local linearization is a good approximation, since the effect of the variable coefficients or the nonlinearity of the problem is still approximately preserved in the total region. Indeed, local linearization also is used in FD and FE methods. The problem now has been reduced into one with many finite regions, where analytic solutions can be obtained, if the boundary conditions in each simple finite line element are properly specified.

Let the governing equation in a line element be $L(y(x)) = G$ where L is now a linear second order differential operator, and let y_N, y'_N, y_S, y'_S be the nodal

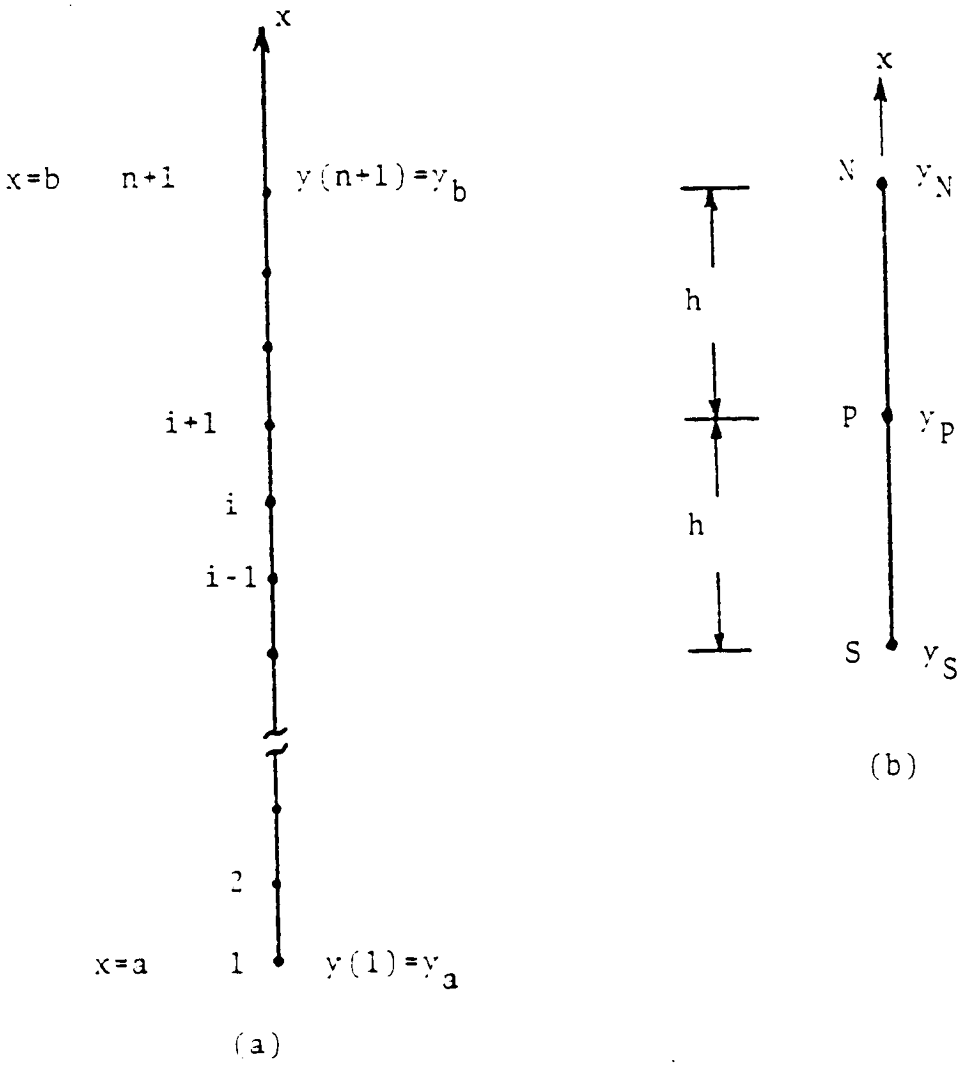

Figure III-1. The Region Under Consideration

(a) The Whole Region

(b) Typical Line Segment

value and its derivative at the northern and southern boundary of the line element. The analytic solution can be obtained anywhere in the line as a function of the boundary conditions

$$y = f(y_N, y_S, y'_N, y'_S, h, x, G) \qquad \text{(III-2)}$$

h is the distance between the midpoint p and the boundary points, s and N. When equation (III-2) is evaluated at the point p, it provides an analytic relationship between the functional value at the interior point p of the local subregion y_p, and its surrounding points N and S or

$$y_p = f(y_N, y_S, y'_N, y'_S, h) \qquad \text{(III-3)}$$

Furthermore, since Equation (III-2) is analytic, it is differentiable. Thus differentiating Equation (III-2) and evaluating at the point p, we have

$$y'_p = f'(y_N, y_S, y'_N, y'_S, h) \qquad \text{(III-4)}$$

Equations (III-3) and (III-4) are the fundamental formulae for the present FA method. For the linear or locally linearized problem, the 3-point FA formula has the form

$$\begin{aligned} y_p &= C\, y_S + C'_N y_N + D\, y'_S + D_N y'_N \\ y'_p &= C'_S y_S + C'_N y_N + D'_S y'_S + D'_N y'_N \end{aligned} \qquad \text{(III-5)}$$

where the coefficients C, D, C', D' are obtained from the local analytic solution. It should be noted here that the finite analytic solution obtained in Equation (III-5) in the interior of the subregion is exact in the sense that it is obtained from an analytic solution to the ODE in the finite subregion. The only approximation involved, if any, is from the local approximation made on the coef- ficient or, nonlinear term of the governing equation.

In an internal finite subregion of the total region D, the neighboring nodal value of y_N, y_S, y_N', y_S' are, in general, unknown. However, they can be in turn expres- sed as an analytic function of their neighboring points. This procedure may be repeated for all the unknown nodes (1) in the total region D. Thus, in general,

$$y_i = C_{i-1}y_{i-1} + C_{i+1}y_{i+1} + D_{i-1}y_{i-1}' + D_{i+1}y_{i+1}'$$
$$y_i' = C_{i-1}'y_{i-1} + C_{i+1}'y_{i+1} + D_{i-1}'y_{i-1}' + D_{i+1}'y_{i+1}' \qquad \text{(III-6)}$$

where y_i, y_i' are the nodal value and its derivative at the midpoint of a given subregion, and other y's in the Equation (III-6) are the boundary values given in Equa- tion (III-5). The assembly of all the expressions for all nodes can then be expressed in matrix form. The system of algebraic equations can now be solved numeri- cally as in the finite difference method to give the nu- merical solution of the total problem.

There is an essential difference between the FA method just described, and the other numerical schemes, such as the FD method and the FE method. In the FD method, the relation between y_p and its neighboring points y_N, and y_S is not obtained from the analytic solution of the differential equation, but from the difference formula or from the truncated Taylor series expansion of the dependent variable about its neighboring points. On the other hand, the FE method assumes an approximated functional form, normally some polynomial of a lower degree, say up to the 5^{th} or 6^{th} degree to represent the solution and uses the variational or Galerkin type of integration on the differential equation to find the relation between y_p and its neighboring points y_N, and y_S.

In the following chapters, some typical second order differential equations will be treated. Examples are solved to illustrate the detailed solution procedure of the FA method.

CHAPTER IV

FINITE ANALYTIC METHOD FOR A SECOND ORDER BOUNDARY VALUE PROBLEM OF NONLINEAR DIFFERENTIAL EQUATIONS

In this chapter, the FA method will be applied to the boundary value problem of a nonlinear second order ordinary differential equation.

IV-1. Derivation of FA Formula

Let us consider the nonlinear ordinary differential equation of the form

$$y'' + A(y',y,x)\, y' + B(y',y,x)\, y = C(y',y,x)$$

$$a < x < b \qquad \text{(IV-1)}$$

Subject to the boundary conditions

$$y(a) = \alpha_1 \text{ and } y(b) = \alpha_2 \qquad \text{(IV-2)}$$

If Equation (IV-1) is nonlinear or linear, but with variable coefficients, then an analytic solution of Equations (IV-1) and (IV-2) is difficult to obtain. A numerical solution is then sought. The first step in applying the FA method is to subdivide the total region into small subregions as shown in Figure (IV-1a). If the finite subregions are small enough, the nonlinear term or the

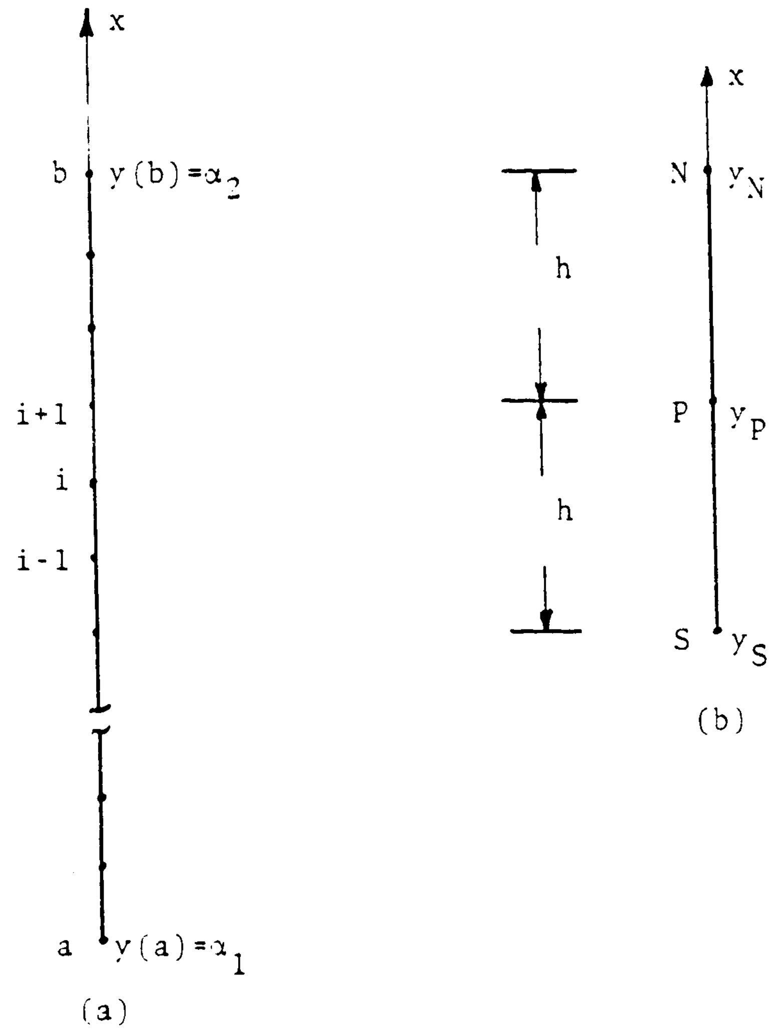

Figure IV-1 Problem Domain

(a) Total Region of the Problem

(b) A Typical Element

variable coefficients can be made constant locally. Thus, Equation (IV-1) can be written as:

$$y'' + Ay' + By = C \tag{IV-3}$$

where A, B and C are constants for each finite subregion of 2h length. The solution of the Equation (IV-3) can be readily obtained (21). The boundary conditions for a typical element can be written as:

$$\begin{aligned} y(o) &= y_S \\ y(2h) &= y_N \end{aligned} \tag{IV-4}$$

Depending on the magnitude of $A^2 - 4B$, three different cases of solution can be realized as follows:

I. $A^2 - 4B < 0$

In this case, the characteristic equation

$$n^2 + A_m + B = 0 \tag{IV-5}$$

has two imaginary and distinct roots, $p \pm iq$, and the solution is:

$$y = e^{px} [C_1 \text{ Cos } qx + C_2 \text{ Sin } qx] + \frac{C}{B} \tag{IV-6}$$

where $p = -\frac{A}{2}$

$$q = \frac{4B - A^2}{2} \tag{IV-6-1}$$

II. $A^2 - 4B = 0$

In this case, the characteristic Equation (IV-4) has two real and equal roots, $m_1 = m_2 = m$, and the solution is:

$$y = (C_1 + C_2 X) e^{mx} + \frac{C}{B} \qquad \text{(IV-7)}$$

where $m = \frac{-A}{2}$ (IV-7-1)

III. $A^2 - 4B < 0$

In the third case, the characteristic Equation (IV-4) has two real and distinct roots m_1 and m_2, and the solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \frac{C}{B} \qquad \text{(IV-8)}$$

where

$$m_1 = \frac{-A + A^2 - 4B}{2}$$

$$m_2 = \frac{-A - A^2 - 4B}{2} \qquad \text{(IV-8-1)}$$

C_1 and C_2 are constants to be determined using boundary conditions (IV-4) for each finite element of length 2h. Now, the FA formula for a typical element for the above cases will be found.

Case I.

$$y = e^{px}[C_1 \text{Cos } qx + C_2 \text{Sin } qx] + \frac{C}{B} \qquad \text{(IV-9)}$$

boundary conditions.

$$y(0) = y_S \qquad\qquad y(2h) = y_N$$

Substituting the boundary conditions into Equation (IV-6), we have: $C_1 = y_s - \frac{C}{B}$

$$C_2 = \frac{y_N - y_s \; e^{2ph}\text{Cos } 2qh + \frac{C}{B}\,(e^{2ph}\text{Cos } 2qh - 1)}{e^{2ph}\text{Sin } 2qh} \qquad \text{(IV-10)}$$

substituting Equations (IV-10) into Equation (IV-6), one can find the analytic solution in the finite sub-region in terms of the nodal value of y at the boundaries of the finite line element. In particular, for y at point p $(x=h)$ y_p, Equation (IV-6) reduces to the following algebraic form:

$$y_p = \left(\frac{e^{ph}}{2\text{ Cos } qh}\right) y_S + \left(\frac{e^{-ph}}{2\text{ Cos } qh}\right) y_N - \frac{C}{B}\left(\frac{e^{ph}}{2\text{ Cos } qh} + \frac{e^{-ph}}{2\text{ Cos } qh} - 1\right) \qquad \text{(IV-11)}$$

which can be written as

$$y_p = C_S y_S + C_N y_N + C_p \qquad \text{(IV-12)}$$

where

$$C_S = \frac{e^{ph}}{2\text{ Cos } qh}, \quad C_N = \frac{e^{-ph}}{2\text{ Cos } qh}, \quad C_p = \frac{C}{B}\,(C_S + C_N - 1) \qquad \text{(IV-13)}$$

Repeating the above procedure for the other two cases we obtain the FA solution with similar forms as Equation (IV-12), but for case II, $(A^2 - 4B = 0)$

$$C_1 = y - \frac{C}{B}$$

$$C_2 = \frac{y_N - y_S e^{2mh} - \frac{C}{B}(1 - e^{2mh})}{2h\, e^{2mh}} \qquad \text{(IV-14)}$$

and therefore

$$C_S = \frac{e^{mh}}{2}, \quad C_N = \frac{e^{-mh}}{2}, \quad C_p = \frac{C}{B}(C_s + C_N - 1) \qquad \text{(IV-15)}$$

for the third case $(A^2 - 4B > 0)$

$$C_1 = \frac{y_S e^{2m_1 h} - y_N + \frac{C}{B}(1 - e^{2m_2 h})}{e^{2m_2 h} - e^{2m_1 h}}$$

$$\text{(IV-16)}$$

$$C_2 = \frac{y_N - y_S e^{2m_1 h} - \frac{C}{B}(1 - e^{2m_1 h})}{e^{2m_2 h} - e^{2m_1 h}}$$

and thus

$$C_S = C_S = \frac{e^{(m_1 + m_2)h}}{e^{m_1 h} + e^{m_2 h}}, \quad C_N = \frac{1}{e^{m_1 h} + e^{m_2 h}},$$

$$\text{(IV-17)}$$

$$C_p = \frac{C}{B}(C_S + C_N - 1)$$

As can be observed from the above equations, the general form of the solution is:

$$C_S y_S - y_p + C_N y_N = C_p \qquad \text{(IV-13)}$$

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or more generally

$$C_{i-1}y_{i-1} - y_i + C_{i+1}y_{i+1} = C_i \qquad \text{(IV-19)}$$

This equation is the finite analytic (FA) representation of the original problem (IV-1), (IV-2), and can be solved numerically by elimination or iterative methods.

IV-2. Calculation of Derivatives

If the functions $A(y',y,x)$ and $B(y',y,x)$ in Equation (IV-1) involve the derivative of the dependent variable y', the derivative of y can be found simply by differentiating the local analytic solution (IV-6) to (IV-3). But the problem is still nonlinear; therefore, the solution procedure for the FA method involves an iterative scheme. That is, it is necessary to renew the value of the derivative y' as well as the function y in A and B for each iteration until the difference of FA solution for y between two iterations is small enough. To linearize $A(y',y,x)$ and $B(y',y,x)$ with average value of y and y' over each finite subregion requires the analytic solution of the derivative y' in addition to y. After local linearization of the coefficients A, B and C, the analytic solution is found for Equation (IV-5). Derivatives of y can be obtained easily by differentiating Equations (IV-6) to (IV-8), or from Equation (IV-6).

$$y' = pe^{px}[C_1 \text{Cos } qx + C_2 \text{Sin } qx]$$

$$+ e^{pn}[-qC_1 \text{Sin } qn + qC_2 \text{Cos } qn] \qquad \text{(IV-20)}$$

and from Equation (IV-8)

$$y' = C_1 m_1 e^{m_1 x} + C_2 m_2 e^{m_2 x} \qquad \text{(IV-21)}$$

The constants C_1 and C_2 for case I are given in Equation (IV-9), for case II in Equation (IV-14), and for Case III in Equation (IV-16).

Since the approximation is over the interval, the average value of y and y' should be found for each finite subregion. For this purpose, one can use the Simpson's integration formula i. e.

$$\text{average } y = \frac{\int_S^N y dx}{2h}$$

$$= \frac{\frac{h}{3}(y_S + 4y_p + y_N)}{2h}$$

$$= \frac{1}{6}(y_S + 4y_p + y_N) \qquad \text{(IV-22)}$$

Similarly for the derivative

$$\text{average } y' = \frac{1}{6}(y'_S + 4y'_p + y'_N) \qquad \text{(IV-23)}$$

IV-3. Calculation of Derivatives at the Boundaries $x = a$, $x = b$

Equations (IV-20) to (IV-22) give the value of derivative of the function in every point in the sub-region of length 2h. The nodal values of the derivative of the function are needed at the beginning of each iteration. For $x = a$ and $x = b$, the same equations are used, but the coefficients of the equations would be the same as the coefficients of their neighboring points because they both belong to the same interval. Therefore, we can write the solution as follows:

Case (I)

$$y'(a) = pe^{pa}[C_1 \text{Cos } qa + C_2 \text{Sin } qa]$$

$$+ e^{pa}[-qC_1 \text{Sin } qa + qC_2 \text{Cos } qa]$$

Case (II)

$$y'(a) = C_2 e^{ma} + (C_1 + C_2 a) me^{ma}$$

Case (III)

$$y'(a) = C_1 m_1 e^{m_1 a} + C_2 m_2 e^{m_2 a}$$

The same procedure may be used for $x = b$ as shown in Figure (IV-2).

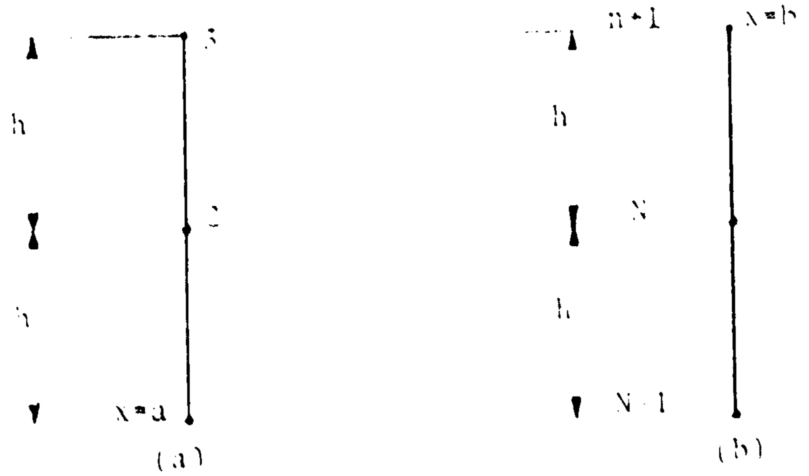

Figure IV-2. First and Last Intervals
(a) First Interval
(b) Last Interval

Procedure for FA solution of Equation (IV-1), therefore, consists of the following steps:

Step I. Local Linearization

The first step of the FA method is to subdivide the problem region into many subregions. In each region, the nonlinear terms are linearized and then each term is approximated

The constant represents an average of the variable coefficient in the subregion. The FA solution given in Equation (IV-19) is used in the calculations.

Step 2. The Initial Profile

As shown in Equation (IV-19) the FA method is an implicit method. If the equation is nonlinear, an initial guessed solution is needed to start the iteration, so that a better approximation for the nonlinear terms can be made. One simple choice for the initial iteration is the line joining the two boundary points y(a), y(b) at two ends of the total region (a,b). Another choice is a second order polynomial passing through the two end points and one mid point.

Step 3. Coefficient Tabulation

This step is to find the FA coefficients given in Equation (IV-19) using Equations (IV-13) to (IV-17).

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Step 4. FA Solution

In this step, the system of linear algebraic Equations (IV-19) is solved by the elimination

Step 5. Iteration

The new and old values of the function at each node are now compared. If the discrepancies are in the desired range, the converged solution is obtained. If not, the procedure is repeated again from Step 2. But, instead of using the initial profile, the calculated nodal values of the function and its derivative are used as new values to evaluate A and B. At this stage, if needed, the following over (under) relaxation parameter may be used. Let $\bar{y}_{j+1}$ and $\bar{y}'_{j+1}$ be the new nodal value of the function and its derivative just obtained from the calculation. Then we have:

$$y_{j+1} = y_j + \gamma(\bar{y}_{j+1} - y_j)$$

$\gamma > 1$ over relaxation

$\gamma < 1$ under relaxation

$j = 0,1,2.$ is the iteration index

Similarly , the over and under relaxation scheme for the derivative can be written as

$$y'_{j+1} = y'_j + \gamma(\bar{y}'_{j+1} - y_j)$$

where y_{j+1} and y'_{j+1} are over relaxed or under relaxed values of the function and its derivative to be used in the next calculation. The flow chart for the above five steps is shown in Figure (IV-3).

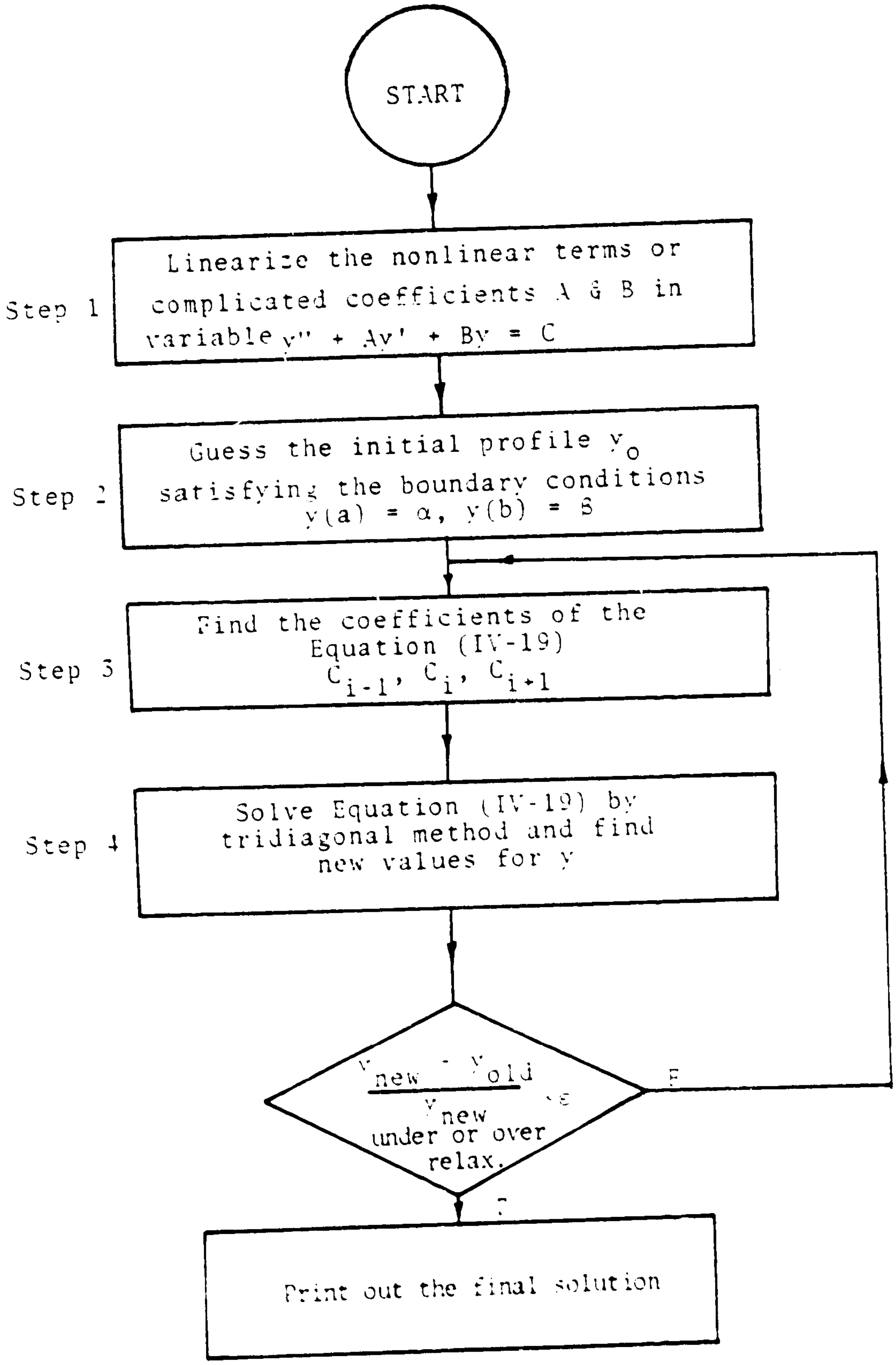

Figure IV-3. FA Flow Chart

CHAPTER V

ILLUSTRATIVE EXAMPLES

In this chapter, some examples of FA solutions are considered. All examples chosen have a known solution so that the FA solutions may be compared not only with the finite difference solution or other numerical solutions, but also with the exact solution.

V-1. Linear Equation with Variable Coefficients

Consider the ordinary differential equation

$$y'' + 4xy' + 2(1+2x^2)\, y = 0 \tag{V-1}$$

subject to boundary conditions

$$\begin{aligned} x &= 0 \qquad y = 0 \\ x &= 1 \qquad y = 1 \end{aligned} \tag{V-2}$$

The analytic solution for this equation is $y = xe^{-x^2+1}$. In order to apply the FA method to Equation (V-1), it is rewritten in the standard FA form as given in Equation (IV-3), where the values of A, B and C for each element of length 2h is obtained by taking the integral average of the variable coefficient over that interval respectively. For example, when x_i is the center node of the finite element,

$$A = \frac{\int_{x_{i-1}}^{x_{i+1}} 4x\,dx}{2h}$$

Since three nodal values are available, a Simpson's closed interval formula gives:

$$A = \frac{1}{6}\,(A_{i-1} + 4A_i + A_{i+1})$$

where $A_i = 4x_i$

Similarly, for B

$$B = \frac{\int_{x_{i-1}}^{x_{i+1}} 2(1+2x^2)\,dx}{2h} = \frac{1}{6}\,(B_{i-1}+4B_i+B_{i+1})$$

where $B_i = 2(1+2x_i^2)$

The coefficient C is zero in this example.

Once the values of A, B and C are determined for each interval of length 2h, the coefficients of the finite analytic equation (IV-17) can be obtained using Equations (IV-6) to (IV-16) as

$$C_{i-1}y_{i-1} - y_i + C_{i+1}y_{i+1} = C_i \qquad \text{(IV-19)}$$

The system of algebraic equations (IV-19) can now be solved numerically by elimination method to provide the FA solution of Equation (V-1).

Table (V-1) shows the FA solution and the numerical solutions obtained from FD and shooting methods in addition to the analytic solution. All numerical calculations in the table (V-1) are made with an increment

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Table V-1

Numerical Solutions of Equation (V-1) VS the Analytic Solution

x	Exact	FD	FA	Shooting
0.0	0.0	0.0	0.0	0.0
0.1	0.26912	0.26918	0.26913	0.26912
0.2	0.52233	0.522450	0.52236	0.52234
0.3	0.74529	0.74554	0.74532	0.74530
0.4	0.92654	0.92673	0.92658	0.92655
0.5	1.05850	1.058699	1.05853	1.05850
0.6	1.13788	1.138081	1.13792	1.13789
0.7	1.16570	1.16587	1.16573	1.16571
0.8	1.14666	1.14678	1.14668	1.14668
0.9	1.08832	1.038839	1.08833	1.08833
1.0	1.0000	1.0	1.0	1.0
Time Used		5.537SRU	7.854SRU	8.123SRU

interval of h = 0.02 which gives 50 intervals in the solution domain. The FD solution with 50 intervals is also obtained using the following finite difference equation:

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} + 4x_i \frac{y_{i+1}-y_{i-1}}{2h} + 2(1+2x_i^2)\, y_i = 0$$

The shooting method solutions using a fourth order Runge-Kutta integration scheme are also obtained in Table (V-1). The tabulated values are after 10 shootings.

Comparison of different solutions in Table (V-1), shows that the finite analytic solution is definitely better than the finite difference solution, while the shooting method, which is based on integration of the equation using an accurate fourth order Runge-Kutta algorithm, gives slightly better solutions than the finite analytic solution. However, it is found that if the missing initial condition is changed slightly, the solution does not converge to the exact solution. Also, since an initial condition has to be guessed, the problem involves iteration. For this problem with the exact missing condition, convergence is achieved after 10 iterations. For the interval size of h = 0.02, the time used in three cases are also listed in the table. The procedure for finding the missing initial conditions starts with guessing two different initial conditions and finding their corresponding boundary values. If these values are different from the prescribed boundary condition, another initial condition is guessed using linear interpolation. This procedure can be repeated by using another linear or perhaps a quadratic or higher order interpolation to produce a sequence of new values for the missing initial condition until a selected assumed value of the initial condition produces the boundary value solution as accurately as desired.

Table (V-2) shows the effect of grid size on the accuracy of the FD and FA solutions, and Table (V-3) is an indication of the error of the predicted solution at $x = 0.4$, produced by each method. Note that for small grid size, both methods yield good results. However, as the grid size becomes larger, the finite difference solution shows more error than the FA solution. This example shows that the FA solution is less sensitive to the interval size. All three methods used for this problem are stable, which is due to the linearity of the equation.

V-2. Nonlinear Ordinary Differential Equation I

As a second example, we consider the following nonlinear differential equation:

$$yy'' + y'^2 = 0 \qquad \text{(V-3)}$$

Table V-2

Comparison of Numerical Solution of
Equation (V-1) using the FD and FA Methods

x \ h	0.02	0.1	0.2	0.5	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.5224	0.5251	0.5336		0.5223
0.4	0.9267	0.9312	0.9456		0.9265
0.5				1.2	1.0585
0.6	1.1380	1.1427	1.1578		1.1378
0.8	1.1467	1.1497	1.1593		1.1466
1.0	1.0	1.0	1.0	1.0	1.0

Finite Difference

x \ h	0.02	0.1	0.2	0.5	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.5223	0.5229	0.5246		0.5223
0.4	0.9265	0.9274	0.9300		0.9265
0.5				1.0821	1.0585
0.6	1.1379	1.1387	1.1412		1.1398
0.8	1.1455	1.1471	1.1485		1.1466
1.0	1.0	1.0	1.0	1.0	1.0

Finite Analytic

Table V-3

Comparison of % error of the Numerical Solution of Equation (V-1) for FA and FD Methods at Point x=0.4

h	0.02	0.1	0.2	0.5
	% Error	% Error	% Error	% Error
FA	0.00	0.114	0.44	2.22
FD	0.019	0.536	2.16	13.42

subject to boundary conditions

$$y(0) = 0 \qquad y(2) = 2 \tag{V-4}$$

We note that Equation (V-3) is singular at $x = 0$ because of the boundary condition $y(0) = 0$.

V-2-1. FA Solution

In order to apply the FA method, the nonlinear terms Equation (V-3) are first linearized in a finite interval of length 2h by its integral average as

$$\bar{y}y'' + \bar{y}'y' = 0 \tag{V-5}$$

where

$$\bar{y}' = \frac{\int_{x_{i-1}}^{x_{i+1}} y'dx}{2h}$$

and

$$\bar{y} = \frac{\int_{x_{i-1}}^{x_{i+1}} ydx}{2h}$$

The linearization in effect eliminates the singularity at $x = 0$ since the integral average of y has replaced the function y in the first 2h interval of Equation (V-2). Equation (V-3) can now be written as:

$$y'' + \frac{\bar{y}'}{\bar{y}} y' = 0 \tag{V-6}$$

Comparing Equation (V-6) with the FA standard form Equation (IV-3), we have

$$A = \frac{\bar{y}'}{\bar{y}}, \; B = 0.0, \; C = 0.0 \tag{V-7}$$

To find the finite analytic solution of Equation (V-7), an iteration between the function y in the linearized coefficient A given in Equation (V-7) and the solution must be made. To conduct an iterative procedure, an initial profile for y over the whole region is first required. As mentioned in Chapter IV, a simple choice for the initial profile is the profile that satisfies the boundary condition at $y(0) = 0$, and $y(2) = 2$, e.g. the line $y = x$. The initial profile for y' is obtained by differentiating the initial profile for y, that is

$$y' = 1$$

Once the coefficients A, B, and C are known for each finite subregion, the locally linearized differential equation can be solved analytically for each interval. Once the new or improved solution for y and y' at each node are obtained, the values of $\bar{y}$ and $\bar{y}'$ for each interval of length $2h$ can be updated using Equations (IV-22), (IV-23) or

$$\bar{y}_i = \frac{y_{i-1} + 4y_i + y_{i+1}}{6}$$
$$\bar{y}'_i = \frac{y'_{i-1} + 4y'_i + y'_{i+1}}{6} \qquad \text{(V-8)}$$

where $\bar{y}_i$, $\bar{y}'_i$ are the average values of the function and its derivative over an interval of length $2h$ where the midpoint is the ith node. Substitution of Equation (V-8) into Equation (V-5) gives the new iterative values for A, B, and C. This procedure is repeated until the convergence is achieved in a desired range.

The analytic solution for Equation (V-4) is

$$y^2 = 2x \tag{V-9}$$

differentiating Equation (V-9)

$$y' = \frac{1}{2x}$$

which is infinity x = 0. Therefore, the shooting method cannot be used for this problem because, no matter how large the missing initial condition is taken, it will never converge to the exact value. Thus, for this problem the FA solution will be compared with the FD solutions (Tables (V-5 and (V-6)) and the exact solution. Table (V-4) shows the results for h = 0.05. The number of iterations for both methods is 8.

V-2-1. Interval Average approximation VS Quasi-linearization

This technique which is used for linearizing the nonlinear terms of the differential equation is based on replacing the nonlinear terms by the integral average of their values over each finite subregion. However, as mentioned in Chapter II, the quasilinearization technique consider the nonlinear second order ordinary differential equation

$$y'' = f(y, y') \tag{V-10}$$

The function f can be expanded in Taylor series around

Table V-4

Numerical Solutions of Equation (V-3) VS

the Exact Solution with 0.05

x	Exact	FA h=0.05	FD h=0.05
0.0	0.00	0.00	0.00
0.2	0.632	0.629	0.638
0.4	0.894	0.892	0.898
0.6	1.095	1.094	1.098
0.8	1.264	1.264	1.266
1.00	1.414	1.413	1.415
1.2	1.549	1.548	1.550
1.4	1.673	1.673	1.674
1.6	1.788	1.788	1.789
1.8	1.897	1.897	1.897
2.00	2.000	2.000	2.000
No. of Iterations		3	8

Table V-5

Comparison of Numerical Solution of Equation (V-3) using the FD and the FA Methods with Varying Grid Sizes

x \ h	0.05	0.1	0.2	0.4	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.4	0.898	0.901	0.907	0.923	0.894
0.8	1.266	1.268	1.272	1.277	1.264
1.2	1.550	1.551	1.553	1.556	1.549
1.6	1.789	1.789	1.790	1.792	1.788
2.0	2.00	2.00	2.00	2.00	2.00
No. of Iteration	8	8	7	5	

Finite Difference

x \ h	0.05	0.1	0.2	0.4	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.4	0.892	0.890	0.886	0.870	0.894
0.8	1.264	1.263	1.261	1.256	1.264
1.2	1.548	1.548	1.547	1.544	1.549
1.6	1.788	1.788	1.788	1.787	1.788
2.0	2.0	2.0	2.0	2.0	2.0
No. of Iteration	8	-	6	5	

Finite Analytic

Table V-6

Comparison of % Error of the Numerical Solution of Equation (V-3) for the FA and FD Methods at x=0.8

h	0.05	0.1	0.2	0.4
	% Error	% Error	% Error	% Error
FA	0.01	0.08	0.23	0.63
FD	0.15	0.31	0.63	1.03

a given function $y_o(x)$ and its derivativative $y_o'(x)$. Thus,

$$f(y,y') = f(y_o(x),\ y_o'(x)) + (y'(x) - y_o'(x))$$

(V-11)

Substituting Equation (V-11) into Equation (V-10), we have

$$y'' = f(y_o(x),\ y_o'(x)) + (y'(x_o) - y_o'(x))$$

(V-12)

which is a linear equation.

Applying this technique to Equation (V-5),

$$y'' = -\frac{y'^2}{y} = f(y,y')$$

$$y'' = -\frac{y_o'^2}{y_o} + (y'(x) - y_o'\ (x))\ \left(\frac{-2y_o'}{y_o}\right) + (y(x)$$

$$- y_o(x))\ \left(+\frac{y_o'^2}{y_o^2}\right)$$

$$\text{or} \quad y'' = -\frac{2y_o'}{y_o}\, y'(x) + \frac{y_o'^2}{y_o^2}\, (y(x)$$

$$\text{or} \quad y'' + \frac{2y_o'}{y_o}\, y' - \frac{y_o'^2}{y_o^2} = 0$$

for which

$$A = \frac{2y_o'}{y_o}, \quad B = -\frac{y_o'^2}{y_o^2}, \quad C = 0$$

Table (V-7) shows the comparison between quasilinearization and interval average approximation for h = 0.1. The table shows that for this problem the interval average approximation gives more accurate results.

From Table (V-5) we see that the number of iterations are about the same for both finite difference and finite analytic method, and vary between 8 iterations for h = 0.05 to 5 iterations for h = 0.4.

Again, more accurate results are obtained with a smaller grid size. The finite analytic solution again proves to be more accurate than the finite difference solution for all grid sizes as shown in Table (V-6) where the solution is compared at x = 0.8.

V-3. Nonlinear Ordinary Differential Equation II

As the next example, consider another nonlinear differential equation:

$$y'' = e^y \qquad \text{(V-13)}$$

with the boundary conditions

$$y(0) = y(1) = 0 \qquad \text{(V-14)}$$

As before, the nonlinear term must be

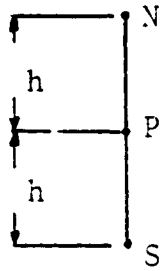
Table V-7

Comparison of Interval Average Approximation (a)

and Quasilinearization (b)

	a	b
x	y	y
0.0	0.0	0.0
0.4	0.890	0.862
0.8	1.263	1.248
1.2	1.548	1.540
1.6	1.788	1.784
2.0	2.00	2.00
Maximum Error	0.22%	3.36%

locally linearized. In order to write Equation (V-13) in the standard form, we consider the small subregion as shown below



In this small subregion, the nonlinear term which is the exponential term, can be expanded about the point (S). Therefore

$$e^{y} = e^{y_p} + (y-y_p)\, e^{y_p} + \ldots\ldots \qquad \text{(V-15)}$$

Approximation of Equation (V-15) is known as quasi-linearization (17) substituting Equation (V-15) into Equation (V-13), one has the locally linearized equation.

$$y'' - e^{y_p} y = e^{y_p}(1-y_p) \qquad \text{(V-16)}$$

Comparing Equation (V-16) with Equation (IV-3) gives

$$A = 0,\ B = -\,e^{y_p},\ C = e^{y_p}(1-y_p)$$

Equations (V-13) and (V-14) have an analytic solution

$$y = -\log 2 + 2\log[C\ \mathrm{Sec}(\tfrac{C}{2}(x-\tfrac{1}{2}))]$$

where $C = 1.3360557$

This equation has been solved by many authors (18), (22). Again, to impliment this iterative procedure, an initial guessed solution is needed. A simple choice could be a polynomial that satisfies the boundary conditions $y(0) = y(1) = 0$ or $y + x(x-1)$. Numerical solutions of (V-13) are compared with the exact solution in Table (V-8).

Both FA and FD methods with a grid size of h give good results after 2 iterations up to 5 decimal points. The time used for both methods is the same (about 3 SRU). But, when the length of the finite subregion is increased, the FA method gives more accurate results than the FD method. In the shooting method, if the exact missing initial condition is not guessed, convergence cannot be achieved. The sensitivity of the solution to the missing initial condition can be demonstrated. For example, even when the missing initial condition is guessed correctly to three decimal points, ten iterations (shooting) is needed before convergence occurs, and the solution still has errors as can be seen from Table (V-8). Generally speaking, the FA method, FD method and shooting method give close results when the step size is small provided a good initial guess is used for the shooting method. The error grows when the step size increases, i.e. less number of points is used. Table (V-9) shows the effect of step size (h) on the numerical solution of Equation (V-13). Table (V-9) shows that for this problem, using the finite difference method, the error grows when the

Table V-8

Comparison of Numerical Solutions of Equation (V-13)

for h=0.02

x	Analytic	FA	FD	Shooting
0.0	0.0000	0.0000	0.0000	0.0000
0.1	-0.04143	-0.04143	-0.04193	-0.04142
0.2	-0.07326	-0.07326	-0.87326	-0.07324
0.3	-0.09580	-0.09579	-0.09579	-0.09575
0.4	-0.109238	-0.109237	-0.10923	-0.10918
0.5	-0.113704	-0.113703	-0.113702	-0.11363
0.6	-0.109238	-0.109237	-0.109234	-0.10915
0.7	-0.09580	-0.09579	-0.09579	-0.09570
0.8	-0.07326	-0.07326	-0.07326	-0.07315
0.9	-0.04143	-0.04143	-0.04144	-0.04130
1.0	0.0000	0.0000	0.0000	0.0000
Iterations or Shooting		2	2	10

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Table V-9

Comparison of Numerical Solutions of Equation (V-13) using the FD, FA and Shooting Methods for Varying Grid Sizes

x \ h	0.02	0.05	0.1	0.2	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.4	-0.1092	-.1092	-.1091	-0.1088	-0.1092
0.8	-0.0732	-0.732	-0.0731	-0.0730	-0.0732
1.00	0.000	0.000	0.000	0.000	0.0

FD Method (2 iterations)

x \ h	0.02	0.05	0.1	0.2	Exact
0.0	0.00	0.00	0.00	0.00	0.0
0.4	-0.1089	-0.1089	-0.1089	-0.1089	-0.1092
0.8	-0.0727	-0.0727	-0.0727	-0.0727	-0.0732
1.0	0.000	0.000	0.000	0.000	0.0

Shooting Method (10 Shootings)

x \ h	0.02	0.05	0.1	0.2	Exact
0.0	0.00	0.00	0.00	0.00	0.0
0.4	-0.1092	-0.1092	-0.1092	-0.1092	-0.1092
0.8	-0.0732	-0.0732	-0.0732	-0.0732	-0.0732
1.00	0.000	0.000	0.000	0.000	0.0

FA Method (2 iterations)

step size becomes larger. The results of the FA method interestingly remains the same for different step sizes as shown in Table (V-9), which shows that the FA solution is insensitive to the step size and gives more accurate solutions than the shooting method. As for the shooting method, an almost exact initial condition must be used, otherwise, the solution is unstable and does not converge to the exact solution. For the FA or FD methods, because of the nonlinearity of the differential equation, the solution procedure requires an iterative process. However, the solution converges with the simple initial profile that is made only to satisfy the boundary conditions. This comparison shows clearly the advantages of the FA method over other methods, especially in the sense of simplicity of the theoretical approach.

Table (V-10) shows the effects of grid size on the error produced by using the FA, FD, and shooting method solutions of Equation (V-9). Again, it is obvious that the FA method produces less error than the FD and shooting solutions. Also, it is interesting that the FA solution for this problem is almost insensitive to the grid size.

Table V-10

Comparison of % Error (at x=0.8) of the Numerical Solution of Equation (V-13) for FA, FD, and Shooting Methods

h	0.02	0.05	0.1	0.2
x	% Error	% Error	% Error	% Error
FA	0.01	0.01	0.01	0.01
FD	0.01	0.015	0.136	0.27
Shooting	0.68	0.68	0.68	0.68

CHAPTER VI

APPLICATION OF FINITE ANALYTIC (FA) METHOD TO FLUID MECHANICS

VI-1. Falkner-Skan Equation

Boundary value problems occur in many fluid mechanics and heat transfer problems. One of the most important problems of this kind is the steady two-dimensional flow of a viscous fluid past a wedge. The problem is to find the velocity profile in the region close to the plate, known as boundary layer (14). The governing equation of the problem is known as the Falkner-Skan equation, which is obtained by similarity transformation from the boundary layer equation, and is given as

$$f''' + ff'' + \beta(1-f'^2) = 0 \qquad \text{(VI-1)}$$

Here f is the dimensionless stream function, derivatives of f are taken with respect to the independent similarity variable η, and β is a parameter of the equation that signifies different flow geometries or pressure gradient exerted on the boundary. $\beta > 0$ denotes the flow is under a favorable pressure gradient and $\beta<0$ under an adverse pressure gradient. The boundary conditions of this flow problem are:

$$\eta = 0,\ f = 0 \qquad \eta = 0,\ f' = 0 \qquad \eta \to \infty,\ f' \to 1 \qquad \text{(VI-2)}$$

VI-2. The FA Solution

For numerical treatment, the infinite boundary condition $\eta \to \infty$ in Equation (V-2) is replaced by a sufficiently large finite boundary $\eta = \eta_\infty$. Then the line η can be subdivided into small line segments (Figure (VI-1)). To implement the FA method, the nonlinear equation (VI-1) is first linearized locally. In order to cast the linearized equation in a form similar to Equation (IV-5), Equation (VI-1) is rewritten in the following form:

$$f''' + f_o f'' - \beta f'_o f' = -\beta \qquad \text{(VI-3)}$$

where f_o and f'_o are the average values of f and f' over the finite subregion of length 2h. Therefore, in the standard form of Equation (IV-5)

$$A = f_o \ , \ B = -\beta f'_o \ , \ C = -\beta \qquad \text{(VI-4)}$$

Let $f' = g$ (VI-5)

Thus, Equation (VI-3) becomes

$$\begin{aligned} & g'' + Ag' + Bg = C \\ & g(0) = 0, \ g(\eta_\infty) = 1 \end{aligned} \qquad \text{(VI-6)}$$

which can be solved numerically for g in each small subregion. Since this problem is nonlinear, the numerical solution again requires an iterative procedure with an initial profile of g satisfying both boundary conditions.

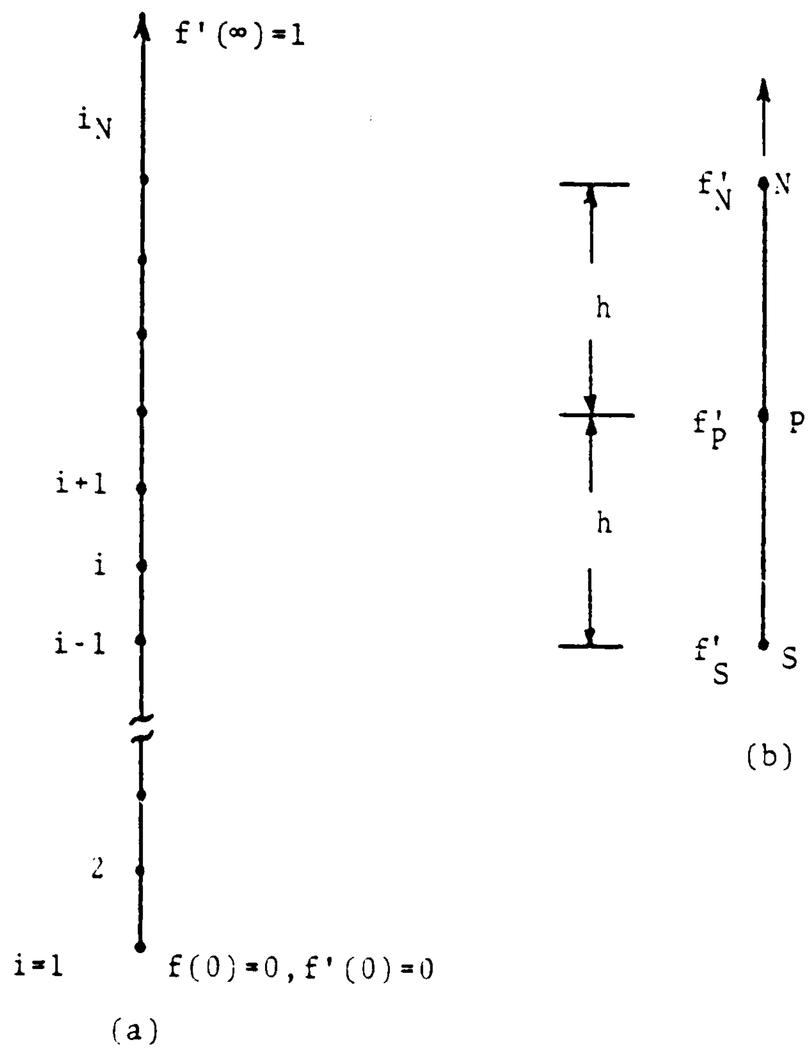

Figure VI-1. Schematic Diagram of Problem

(a) The Whole Region

(b) A Typical Subregion

Once new values g at each nodal point are calculated, the new values B are known for each node. But, for calculation of A, new values for f need to be calculated. From Equation (VI-7) is is obvious that

$$\frac{df}{d\eta} = g \qquad \text{(VI-7)}$$

thus

$$f_p - f_S = \int_S^p df = \int^p gd \qquad \text{(VI-8)}$$

where s and p are southern and middle points of an interval of length 2h, as shown in Figure (VI-1). As discussed in Chapter IV, the solution for dependent variable g in Equation (IV-5) has three different cases according to the value of $A^2 - 4B$. Therefore, for $A^2 - 4B < 0$

$$f_p = f_S + C_1 \frac{e^{ph}}{p^2+q^2} [p\text{Cos } qh + q\text{Sin } qh] +$$

$$+ C_2 \frac{e^{ph}}{p^2+q^2} [p\text{Sin } qh - q\text{Cos } qh] -$$

$$\frac{C_1 p}{p^2+q^2} + \frac{C_2 q}{p^2+q^2} + \frac{C}{B} h \qquad \text{(VI-9)}$$

which is simply the integration of Equation (VI-8). Similarly, for $A^2 - 4B = 0$,

$$f_p = f_S + \left(\frac{C_1}{m_1} - \frac{C_2}{m_2}\right)(e^{mh}-1) + \left(\frac{C_2 e^{mh}}{m} + \frac{C}{B}\right) h$$

(VI-10)

and for $A^2 - 4B > 0$,

$$f_p = f_S + \frac{C_1}{m_1} e^{m_2 h} + \frac{C_2}{m_2} e^{m_2 h} + \frac{C}{B} \qquad \text{(VI-11)}$$

Values of A and B are the average values of initially guessed f and . f' over each finite subregion of length 2h. Therefore, A and B can be obtained using Simpson's integration formula

$$A = \frac{f_S+4f_p+f_N}{6} \quad , \quad B = - \frac{f'_S+4f'_p+f'_N}{6} \qquad \text{(VI-12)}$$

Any initial profile should satisfy both boundary conditions $g(0) = 0$, $g(\eta_\infty) = 1$. If the η_∞ is taken to be 10. a simple initial profile is taken as $g_o = 0.1$.

Thus

$$\int_0^\eta df = \int_0^\eta 0.1\eta d\eta \qquad \text{(VI-13)}$$

or the guessed f profile becomes $f = 0.5\ \eta^2$.

The calculation of the FA method, thus proceeds with Equation (VI-13) as initial guess.

Since there is no analytic solution to Equation (VI-1), the FA solution is compared only with the shooting method, which is the most popular technique for solving the Falkner-Skan flow problems ([illegible]). Table (VI-1) shows the numerical solution of Equation (VI-1) for

Table VI-1

Velocity Profile for the Falkner-Skan Equation using FA Method

η \ β	-0.1988	-0.18	0.00	0.5	1.00
0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.1000	0.2163	0.4609	0.6815	0.7783
2.0	0.3818	0.5617	0.816	0.442	0.973
3.0	0.729	0.860	0.969	0.995	0.998
4.0	0.9404	0.979	0.997	0.999	0.999
5.0	0.994	0.998	0.999	0.999	1.000
6.0	0.999	0.999	1.000	1.000	1.000
7.0	0.999	1.000			
8.0	1.000				
9.0					
10.0					

different values of $\beta(-0.1988<\beta<1)$. Equation (VI-2) shows that the domain of problem is $(0-\infty)$, which cannot be treated by any regular method of solving boundary value problems. In order to satisfy the boundary condition at infinity, it is assumed that for this problem $\eta = \eta_\infty$, $f' \rightarrow 1$. Therefore, the infinite boundary condition is replaced by a finite boundary. For example, $\eta_\infty = 10$, $f' = 1$. The value of 10 may be replaced by other values if the numerical solution does not asymptotically approach $f' \rightarrow 1$. Table (VI-2) shows the comparison between the interval average approximation and quasilinearization. The quasilinearization process for the Falkner-Skan equation can be done as follows:

$$f''' + ff'' + \beta(1-f'^2) = 0$$

$$g'' + f_0 g' + \beta(1-g^2) = 0 \qquad \text{(VI-14)}$$

Therefore, $y(g',g) = -f_0 g' - \beta(1-g^2)$.

Using Equation (V-14), and simplifying

$$g'' + f_0 g' - 2\beta g_0 g = -\beta(1+g_0^2) \qquad \text{(VI-15)}$$

for which $A = f_0$, $B = -2\beta g_0$, $C = -\beta(1+g_0^2)$. As can be seen from Table (VI-2), numerical results are almost identical. However, quasilinearization converges faster than the interval average approximation by an order of 2.

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Table VI-2

Comparison of Numerical Solution of the Falkner-Skan Equation using Shooting and FA Methods for $\beta=0.0$

η \ $\Delta\eta$	0.1	0.5	1.0	1.5	2.0
0.0	0.00	0.00			
2.0	0.816	0.826			
4.0	0.997	1.006			
6.0	0.999	1.009	UNSTABLE......		
8.0	1.000	0.009			
10.0	1.000	0.999			
$f''(0)$	0.4696	0.476			

a. Velocity Profile using Shooting Method

η \ $\Delta\eta$	0.1	0.5	1.0	1.5	2.0
0.0	0.00	0.00	0.00	0.00	0.00
2.0	0.8168	0.8246	0.849		0.9488
4.0	0.997	0.998	0.998		0.999
6.0	0.999	1.000	1.000		1.000
8.0	1.000	1.000	1.000		1.000
10.0	1.000	1.000	1.000		1.000
$f''(0)$		0.4699	0.437	0.499	

b. Velocity Profile using FA Method

VI-3. Numerical Results

The FA numerical results of Table (VI-1) are obtained for h = 0.1. The FA solutions are identical to those obtained by the shooting method to the third digit. Tables (VI-3) and (VI-4) show a comparison of the FA method and the shooting method. The comparison indicates that the FA method is more stable for this problem.

Equation (VI-1) was derived first by Falkner and Skan (23) and was calculated later numerically by Hartree (24). Ever since, because of strong nonlinearity of Equation (VI-1) its solution has been a challenge to many mathematicians as well as engineers. Stewartson (25) found that when $\beta <- 0.1981$, there are two acceptable solutions, one with $f''(0)<0$. In addition, he showed that if $-0.5<\beta<0$, there is a family of solutions corresponding to boundary layer bounded on one side by free streamlines. Later, in 1966, Libby and Liu (26) suggested a point of view and a mechanism making the similarity solutions for $\beta<-0.1988$ physically acceptable, and presented some of the solutions. The numerical analysis that Libby and Liu used is based on the application of the quasilinearization technique developed by Bellman and Kalaba (18) in approximating the governing Falkner-Skan equation. In addition, the boundary condition at infinity is treated by requiring that exponential decay is assured. In their method, instead of specifying β and

Table VI-3

Comparison of Numerical Solution of the Falkner-Skan Equation using Shooting and FA Methods for β=1.0

η \ Δη	0.1	0.5	1.0	1.5	2.0
0.0	0.0	0.0			
1.0	0.7778	0.7769			
2.0	0.9732	0.9722	UNSTABLE......	......	......
3.0	0.9984	0.9980			
4.0	0.9999	0.9999			
6.0	1.0000	1.0000			
f''(0)	1.2326	1.2318			

a. Velocity Profile using Shooting Method

η \ Δη	0.1	0.5	1.0	1.5	2.0
0.0	0.0	0.0	0.0	0.0	
1.0	0.7783	0.7916	0.838		
2.0	0.9733	0.9773	0.9897		
3.0	0.9984	0.9988	0.999	0.9963	
4.0	0.9999	0.9999	1.000		
6.0	1.000	1.000	1.000	1.000	
f''(0)	1.2345	1.2917	1.4994	1.8707	

b. Velocity Profile using FD Method

Table VI-4

Comparison of Quasilinearization and Interval

Average Method for $\beta=0$, $\Delta\eta=0.1$

η	Quasi	Average
0.0	0.0	0.0
1.0	0.4609	0.4609
2.0	0.816	0.816
3.0	0.969	0.969
4.0	0.9977	0.9977
5.0	0.999	0.999
6.0	1.000	1.000
7.0	1.000	1.000
8.0	1.000	1.000
Number of Iterations	4	8

Velocity Profile

seeking $f''(0)$ so that $f'(\infty) = 1$, $f''(0)$ is specified and β is considered as a parameter to be determined in each iteration cycle of the quasilinearization scheme.

In the FA method, we have not imposed any condition in the exponential behavior as $\eta \to \infty$. For the values of $\beta > -0.1988$, replacing the boundary conditions at infinity by a finite large distance from the wall seems to be satisfactory, and yields good results. However, for $\beta < -0.1988$, this substitution is unlikely to succeed, and the solutions obtained do not behave exponentially, i.e. the numerical scheme works as if the outer boundary were a fixed wall. The problem with a boundary condition at infinity was studied by Robertson (27), who considered the linear two point boundary value problem on an infinite interval. In his study, a numerical method, using a finite difference approximation to the second order differential equation is given which tests the suitability of the finite point chosen to represent infinity.
In Robertson's study, the length of the finite interval is calculated such that the replacement of this finite interval for the infinite interval would give solutions with desired accuracy. However, the analysis is only for linear equations. For nonlinear problems, one has to examine the existence and uniqueness of the solution, a subject which has not been fully developed yet.
Keller (28) is studying the existance and uniqueness of the

solution of two point boundary value problems $L(y) = -y'' + p(x)y' + q(x)y = f(x)$ has proved that the existence and uniqueness is guaranteed only if p, q, f are continuous with $q > 0$. Therefore, the Falkner-Skan problem, even when it is linearized, may not have a solution if the parameter β is such that the value of q is not positive. The application of the FA method to the Falkner-Skan problem for values of β must be studied carefully, and can be a subject for further investigation. For $\beta \geq 0$, the FA solution produces satisfactory solution.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

In the present work, the idea of the finite analytic method introduced by Chen and Li, (4) is developed and extended to the solution of linear and nonlinear two point boundary value problems. In general, the FA method is better than the finite difference method for examples treated. In particular, the FA method has the following advantages: it is relatively insensitive to the grid size, more accurate since truncation errors are eliminated or minimized, and more stable. In addition, the FA method, because of its continuous functional solutions in the finite subregion is differentiable. This is a great advantage over other methods, since approximation of derivatives by finite difference or finite element formulae, in general, introduce additional errors in addition to the errors already made in the solution.

In the case of nonlinear boundary value problems, since the equation is nonlinear, both FA and FD methods require linearization. In the present study, the nonlinear term has been replaced by a constant equal to its integral average in each finite subregion. If the finite subregion is small, this approximation is indeed very good. For large subintervals, the approximation will produce some error. However, it is generally less than the error produced by a finite difference method. On the other hand, any second order nonlinear ordinary differential equation can be locally linearized. In the FA method, they are locally cast into a linear second order equation with constant coefficients. Therefore, the FA solution does not require much analytical work and can be implimented easily. Replacing the nonlinear term by a constant is the simplest kind of approximation. Obviously, this approximation can be improved by using a polynomial of arbitrary degree as in approximation of the function in each finite subregion. This will improve the accuracy of the FA solutions, but requires more analytical work, and could be a subject for future development of the FA method. A very important feature of the FA method is its stability as compared to the FD method and shooting method. This advantage can be seen in Chapter VI where a comparison is made between the FA method and shooting method for different grid sizes.

The implimentation of the FA method involves a relatively simple numerical algorithm compared with that used in existing methods in solving boundary value problems. The principle of the FA method is very simple and the analytic part of the method consists only of solving a linear second order ordinary differential equation with constant coefficients.

APPENDIX

THE FINITE ANALYTIC SOLUTION OF A NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

```
PROGRAM SKAN2(INPUT,OUTPUT)
C ***************************************************************
C                         IN THE NAME OF GOD
C                         ******************
C                            ************
C                               ******
C SHEIKHOLESLAMI,MOHAMMAD,ZAHED ,ENERGY DIVISION ,ENGINEERING
C                 DEPARTMENT ,THE UNIVERSITY OF IOWA
C                     IOWACITY ,IOWA ,MAY 1980
C +++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C FINITE ANALYTIC (FA) SOLUTION OF A NON-LINEAR TWO-POINT BOUN-
C DARY VALUE PROBLEM OF ORDINARY DIFFERENTIAL EQUATION........
C ***************************************************************
C THE FINITE ANALYTIC (FA) METHOD IS APPLIED TO A NON-LINEAR
C SECOND ORDER ORDINARY DIFFERENTIAL EQUATION;
C
C           G" +A(G',G,X)G' +B(G',G,X)G =C(G',G,X)
C SUBJECT TO
C            G(A)=D1 ,  G(B)=D2        A<X<B
C
C SOLUTION PROCEDURE STARTS WITH LINEARIZING THE NONLINEAR TERMS
C OF THE ABOVE EQUATION.  THIS IS DONE BY USING THE AVERAGE
C VALUES OF A,B,AND C OVER EACH FINITE SUBREGION. THIS WILL
C CONVERT THE EQUATION TO A LINEAR ORDINARY DIFFERENTIAL
C EQUATION WITH CONSTANT COEFFICIENTS. THIS LINEAR EQUATION
C IS THEN SOLVED ANALITICALLY..................................
C ***************************************************************
C              ****************************************
C                       ***********************
C  PROGRAM SPECIFICATIONS..........
C  ETA : INDEPENDENT VARIABLE
C  G,GO,GA : NEW,OLD,AND AVERAGE VALUES OF THE DEPENDENT
C  VARIABLE ,G
C  F,FO,FA : NEW,OLD,AND AVERAGE VALUES OF THE DERIVATIVE
C  OF G.
C  FP,FPO,FPA : NEW,OLD,AND AVERAGE VALUES OF THE INTEGRAL
C  OF G OVER AN INTERVAL.
C  PM,PM1,PM2 : ROOTS OF THE CHARACTERISTIC EQUATION
C  P,Q : REAL AND IMAGINARY PARTS OF THE ROOTS OF THE
C  CHARACTERISTIC EQUATION WHEN A**2-4*B IS NEGATIVE.......
C  AC1,AC2 : CONSTANTS OF INTEGRATION......................
C  CC1,CC2 : AS DEFINED IN  THE PROGRAM..........
C  ER : ERROR
C  CS,T,CN,CC : COEFFICIENTS OF TRI-DIAGONAL MATRIX......
C  DEL : A**2-4*B
C  ADEL : -DEL
C  GAM : RELAXATION FACTOR.............................
C  DETA : STEP SIZE........
C  NO : NUMBER OF POINTS
C  ******************************************************
C  ++++++++++++++++++++++++++++++++++++++++++++++++++++++
C  ******************************************************
C EQUATIONS USING FINITE.DIFFERENTIAL.METHOD..........
      DIMENSION ETA(201),F(201),FO(201),G(201),GO(201),FA(201),GA(201)
     +,DEL(201),PM1(201),PM2(201),AC1(201),AC2(201),CN(201),CS(201),
     +CC(201),T(201),ER(201),P(201),Q(201),PM(201),ADEL(201),FP(201)
     +,CC1(201),CC2(201),A(201),B(201),C(201),CL(201),FPA(201),FPO(201)
    1 READ*DETA,NO,EPS,KK,GAM,BETA
      NP1=NO
      NO=NO-1
      TR=0.0
```

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```
 CO=1.0
 PRINT 100,DETA,NO,EPS
C *************************************************************
C SET THE INITIAL PROFILE.................*********************
C *************************************************************
 DO 2 I=1,NP1
 ETA(I)=(I-1)*DETA
 G(I)=SQRT(0.1*ETA(I))
 F(I)=0.1
 FP(I)=20*(G(I)**3)/3.
 FO(I)=F(I)
 FPO(I)=FP(I)
 2 GO(I)=G(I)
C *************************************************************
C FIND THE ROOTS OF CHARACTERISTIC EQUATION...............******
C *************************************************************
  10 DO 3 I=2,NO
 FA(I)=(FO(I-1)+4*FO(I)+FO(I+1))/6.
 GA(I)=(GO(I-1)+4*GO(I)+GO(I+1))/6.
 FPA(I)=(FPO(I-1)+4*FPO(I)+FPO(I+1))/6.
 A(I)=FPA(I)
 B(I)=-2*BETA*GA(I)
 C(I)=-BETA*(1+GA(I)**2)
 IF(C(I).EQ.0.0) GO TO 13
 CL(I)=C(I)/B(I)
 GO TO 3
 13 CL(I)=0.0
 3 DEL(I)=A(I)**2-4*B(I)
 DO 41 I=2,NO
  IF(DEL(I)) 29,39,49
C *************************************************************
C COEFFICIENTS OF THE TRI-DIAGONAL MATRIX IF THE CHARACTERISTIC
C EQUATION HAS TWO IMAGINARY AND DISTINCT ROOTS...............
C *************************************************************
 29 ADEL(I)=-DEL(I)
 P(I)=-A(I)/2.
 Q(I)=SQRT(ADEL(I))/2.
 CS(I)=EXP(P(I)*DETA)/(2*COS(Q(I)*DETA))
 CN(I)=EXP(-P(I)*DETA)/(2*COS(Q(I)*DETA))
 GO TO 40
C *************************************************************
C COEFFICIENTS OF THE TRI-DIAGONAL MATRIX IF THE CHARACTERISTIC
C EQUATION HAS TWO REAL AND EQUAL ROOTS........................
C *************************************************************
 39 PM(I)=-A(I)/2.
 CS(I)=EXP(PM(I)*DETA)/2.
 CN(I)=EXP(-PM(I)*DETA)/2.
 GO TO 40
C *************************************************************
C COEFFICIENTS OF THE TRI-DIAGONAL MATRIX IF THE CHARACTERISTIC
C EQUATION HAS TWO REAL AND DISTINCT ROOTS......................
C *************************************************************
  49 PM1(I)=(SQRT(DEL(I))-A(I))/2.
 PM2(I)=(-SQRT(DEL(I))-A(I))/2.
 CN(I)=1./(EXP(PM2(I)*DETA)+EXP(PM1(I)*DETA))
 CS(I)=EXP((PM1(I)+PM2(I))*DETA)*CN(I)
 40 CC(I)=CL(I)*(CS(I)+CN(I)-1)
  T(I)=-1.0
 41 CONTINUE
 GO TO 28
 73 DO 77 I=2,NO
```

```
 CS(I)=EXP(-A(I)*DETA)/(EXP(-A(I)*DETA)+1)
 CN(I)=1./(EXP(-A(I)*DETA)+1)
 T(I)=-1.0
 77 CC(I)=C(I)*DETA*((1-EXP(-A(I)*DETA))*CN(I))/A(I)
 28 CC(1)=GO(1)
 CC(NP1)=GO(NP1)
 CS(NP1)=0.0
 CN(1)=0.0
 T(1)=1.0
 T(NP1)=1.0
C *************************************************************
C ....CALL THE TRI-DIAGONAL SUBROUTINE TO FIND NEW VALUES FOR....
C ......................THE FUNCTION.............................
 CALL TDMX (1,NP1,CS,T,CN,CC,G)
C *************************************************************
C FIND NEW VALUES FOR DERIVATIVES USING NEW VALUES OF THE FUNCTION
C *************************************************************
   DO 42 I=2,NO
 IF(DEL(I)) 27,37,47
C *************************************************************
C  NEW VALUES FOR F IF DEL(I) IS NEGATIVE...................
C *************************************************************
 27 AC1(I)=G(I-1)-CL(I)
 AC2(I)=(G(I+1)-G(I-1)*EXP(2*P(I)*DETA)*COS(2*Q(I)*DETA)
++(EXP(2*P(I)*DETA)*COS(2*Q(I)*DETA)-1)*CL(I))/(EXP(2*P(I)*
+DETA)*SIN(2*Q(I)*DETA))
 F(I)=P(I)*EXP(P(I)*DETA)*(AC1(I)*COS(Q(I)*DETA)+AC2(I)*
+SIN(Q(I)*DETA))+EXP(P(I)*DETA)*(-Q(I)*AC1(I)*SIN(Q(I)*DETA)+
+AC2(I)*Q(I)*COS(Q(I)*DETA))
 GO TO 42
C *************************************************************
C  NEW VALUES FOR F IF DEL(I) IS ZERO........................
C *************************************************************
 37 BC1(I)=G(I-1)-1./GA(I)
 BC2(I)=(G(I+1)-G(I-1)*EXP(2*PM(I)*DETA)-(1-EXP(2*PM(I)*DETA))
+*CL(I))/(2*DETA*EXP(2*PM(I)*DETA))
  F(I)=(BC2(I)+PM(I)*BC1(I)+PM(I)*BC2(I)*DETA)*EXP(PM(I)*DETA)
 GO TO 42
C *************************************************************
C  NEW VALUES FOR F IF DEL(I) IS POSITIVE....................
C *************************************************************
  47 CC1(I)=EXP(2*PM1(I)*DETA)
 CC2(I)=EXP(2*PM2(I)*DETA)
 CP1(I)=(G(I-1)*CC2(I)-G(I+1)-(CC2(I)-1)*CL(I))/(CC2(I)-CC1(I))
 CP2(I)=-(G(I-1)*CC1(I)-G(I+1)-(CC1(I)-1)*CL(I))/(CC2(I)-CC1(I))
 F(I)=CP1(I)*PM1(I)*EXP(PM1(I)*DETA)+CP2(I)*PM2(I)*EXP(PM2(I)*DETA)
 42 CONTINUE
 F(1)=F(2)+BETA*DETA+FPA(2)*G(2)-BETA*GA(2)*FP(2)
 GO TO 79
 78 DO 88 I=2,NO
 AC1(I)=(G(I+1)-G(I-1)-2*C(I)*DETA/A(I))/(EXP(-2*A(I)
+*DETA)-1)
 88 F(I)=-AC1(I)*A(I)*EXP(-A(I)*DETA)+C(I)/A(I)
 F(1)=-AC1(2)*A(2)+C(2)/A(2)
 79 F(NP1)=(G(NP1)-G(NO))/DETA
 DO 52 I=2,NO
 IF(BETA.EQ.0.0) GO TO 35
 IF(DEL(I)) 91,92,93
C ******************************************************
C NEW VALUES FOR FP IF DEL IS NEGATIVE.............
C ******************************************************
```

```
91 FP(I)=FP(I-1)+AC1(I)*EXP(P(I)*DETA)*(P(I)*COS(Q(I)*DETA)+
+Q(I)*SIN(Q(I)*DETA))/(P(I)**2+Q(I)**2)+AC2(I)*EXP(P(I)*DETA)*
+(P(I)*SIN(Q(I)*DETA)-Q(I)*COS(Q(I)*DETA))/(P(I)**2+Q(I)**2)-
+AC1(I)*P(I)/(P(I)**2+Q(I)**2)+AC2(I)*Q(I)/(P(I)**2+Q(I)**2)+
+DETA*CL(I)
GO TO 52
92 FP(I)=FP(I-1)+(BC1(I)/PM(I)-BC2(I)/(PM(I)**2))*(EXP(PM(I)*DETA
+)-1)+DETA*(BC2(I)*EXP(PM(I)*DETA)/PM(I)+CL(I))
GO TO 52
93 FP(I)=FP(I-1)+CP1(I)*EXP(PM1(I)*DETA)/PM1(I)+DETA*CL(I)
++CP2(I)*EXP(PM2(I)*DETA)/PM2(I)-CP1(I)/PM1(I)-CP2(I)/PM2(I)
GO TO 52
35 C2(I)=EXP(2*(-FA(I)*DETA))
FP(I)=FP(I-1)+DETA*((G(I-1)*C2(I)-G(I+1))/(C2(I)-1))+
+(EXP(-FA(I)*DETA)-1)*((G(I+1)-G(I-1))/(C2(I)-1))/(-FA(I))
52 CONTINUE
FP(NP1)=FP(NO)+DETA*G(NO)
C ***FIND THE ERRORS AND PRINT OUT THE FINAL SOLUTION************
DO 6 I=2,NO
6 ER(I)=ABS((GO(I)-G(I))/G(I))
DO 7 I=3,NO
IF(ER(2)-ER(I)) 23,23,7
23 ER(2)=ER(I)
7 CONTINUE
IF(ER(2)-EPS) 33,33,22
22 IF(TR-KK*CO) 15,16,15
15 DO 8 I=1,NP1
GO(I)=GO(I)+GAM*(G(I)-GO(I))
FPO(I)=FPO(I)+GAM*(FP(I)-FPO(I))
8 FO(I)=FO(I)+GAM*(F(I)-FO(I))
TR=TR+1
GO TO 10
16 CO=CO+1
DO 9 I=1,NP1
GO(I)=GO(I)+GAM*(G(I)-GO(I))
FPO(I)=FPO(I)+GAM*(FP(I)-FPO(I))
9 FO(I)=FO(I)+GAM*(F(I)-FO(I))
PRINT 120,(ETA(I),G(I),F(I),I=1,NP1,5)
GO TO 10
33 PRINT 125,(ETA(I),G(I),F(I),I=1,NP1)
PRINT 140,TR
100 FORMAT(5X,*DETA=*,F5.2/5X,*NO=*,I3/5X,*EPS=*,
+F13.6)
110 FORMAT(5X,*ETA*,15X,*GO*,15X,*FO*)
120 FORMAT(5X,F5.2,5X,F13.6,5X,F13.6)
125 FORMAT(5X,F5.2,5X,F13.6,5X,F13.6)
130 FORMAT(15X,*NO SOLUTION*,5X,*DEL=*,F13.6)
140 FORMAT(15X,*TR=*,F4.0//)
GO TO 1
END
C ****************************************************************
C SUBROUTINE TDMX WHICH SOLVES A SYSTEM OF LINEAR EQUATIONS...
C ****************************************************************
SUBROUTINE TDMX(IF,L,A,B,C,D,V)
DIMENSION A(201),B(201),C(201),D(201),V(201),BETA(201),
+GAMMA(201)
BETA(IF)=B(IF)
GAMMA(IF)=D(IF)/BETA(IF)
IFP1=IF+1
DO 1 I=IFP1,L
BETA(I)=B(I)-A(I)*C(I-1)/BETA(I-1)
```

```
1 GAMMA(I)=(D(I)-A(I)*GAMMA(I-1))/BETA(I)
V(L)=GAMMA(L)
LAST=L-IF
DO 2 K=1,LAST
I=L-K
2 V(I)=GAMMA(I)-C(I)*V(I+1)/BETA(I)
RETURN
END
```

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PART II

Numerical Solution of Two-Dimensional Poission And Laplace Equations by Finite Analytic Methods

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LIST OF TABLES

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LIST OF FIGURES

LIST OF SYMBOLS

Alphabetical Symbols

A_{1k}, A_{2k}, A_{3k}	functions (see equation (VI-18))
a	$\frac{(2n-1)\pi}{2}$
a_0, a_1, a_2	coefficients of quadratic expression of the north boundary functions $f_N(x)$
B_{1k}, B_{2k}, B_{3k}	functions (see equation (VI-18))
b	$\frac{n\pi}{2}$
b_0, b_1, b_2	coefficients of quadratic expression of the south boundary function $f_S(x)$
c_0, c_1, c_2	coefficients of quadratic expression of the west boundary functions $f_W(y)$
c_{0i}, c_{1i}, c_{2i}	quadratic expression (see equation (VI-5))
c_{EC}, c_{WC},...	9-point finite analytic formulation coefficients
c_{XEC}, c_{XWC},...	9-point finite analytic formulation coefficients for derivative with respect to x
c_{YEC}, c_{YWC},...	9-point finite analytic formulation coefficients for derivative with respect to y
c_{1EC}, c_{1WC},...	finite analytic coefficients of partial numerical formulas for simpler problems
$c_{i,j}$, $c_{i-1,j}$,...	finite analytic coefficients of the numerical formulas expressed in terms of the nodal points in the total region
$D_n(y)$	coefficients for the Fourier series expansion of the nonhomogeneous term
D_n^{1k}, D_n^{2k}, D_n^{3k}	functions (see equation (VI-17))

ORIGINAL PAGE IS
OF POOR QUALITY

d_0, d_1, d_2	coefficients of quadratic expression of the north boundary function $f_N(x)$
E_{1n}, E_{2n}, E_{3n}, E_{4n}	coefficients for the Fourier series expansion
$f_1(y)$, $f_2(y)$	spline function for the common boundaries equations (VI-7) and (VI-8)
f_N, f_S, f_W, f_E	boundary conditions along boundary lines on north, south, west, and east sides of the subregion
$G_1, \ldots G_{2i}, \ldots, G_{N+1}$	coefficients for finite analytic formula (equation (VI-16))
$G_{y1}, \ldots G_{y2i} \ldots G_{y(N+1)}$	coefficients for derivatives of finite analytic formula (equation (VI-19))
h,k	grid spacing in the x and y direction
L	length scale or partial differential operator
$Q_1, \ldots Q_{2i}, \ldots Q_N$	coefficients for finite analytic formula for region R2, (see equation (IV-20))
$Q_1', \ldots Q_{2i}', \ldots Q_{N+1}'$	coefficient for finite analytic solution for region R2, (see equation (IV-20))
$Q_{y1}, Q_{y1}', \ldots Q_{y(N+1)}'$	coefficients for derivatives of finite analytic formula for region R2, (see equation (IV-21))
$S_1, \ldots S_{2i}, \ldots S_{N+1}$	coefficients for finite analytic formula for region R3, (see equation (IV-22))
$S_{y1}, \ldots S_{y2i}, S_{y(N+1)}$	coefficients for derivatives of finite analytic formula for region R3, (see equation (IV-23))
$W_n(y)$	coefficients for the Fourier series expansion of the nonhomogeneous term in PDE equation (IV-16)
w	relaxation factor used in Chapter V
x,y	independent variables of the PDE

x1,x2,x3	length scale (see Figure (VI-1))
Y1, Y2, Y3	length scale (see Figure (VI-1))

Greek Symbols

∇^2	Laplacian
γ	ratio of grid size, h/k used in finite analytic formulas derived in Chapters IV and V
λ, μ	eigen values
ξ	nonhomogeneous term used in Poisson equation (see equation (IV-1))
π	constant, Pi = 3.14159
ψ	dependent variable of the PDES
ψ_p, ψ_{EC}, ψ_{WC}, ...	are the nodal values of the dependent variable on each corresponding nodal points in the subregion
ψ_x, ψ_y	derivatives of the dependent variable with respect to x and y
ψ_{xx}, ψ_{yy}	second derivative of the dependent variable ψ with respect to x and y
ψ_1, ψ_2, ψ_3, ψ_4	dependent variables of the simpler problems decomposed from a complicated problem
$\psi_{i,j}$, $\psi_{i+1,j}$, ...	the nodal values of the dependent variable at (i,j)th and (i+1,j) nodes
$\psi^m_{i,J}$ ...	mth overall iteration of ψ values in numerical calculations at (i,J) node
$\psi^{m+1}_{i,J}$, ...	m+1 th overall iteration of ψ at (i,J) node
ε	convergence criterion

LIST OF ABBREVIATIONS

ADI	Alternating Direction Implicit
FA	Finite Analytic Method
FD	Finite Difference Method
FE	Finite Element Method
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

CHAPTER I

INTRODUCTION

Many initial value or boundary value problems, in an engineering process may involve complex material properties, complex geometry and boundary conditions and defy the analytic solution. Engineers thus resort to numerical methods to obtain approximate, but acceptable, values of the unknown quantities to a discrete number of points in the region. There are already many established numerical methods available for solving ordinary differential equations (ODE) and partial differential equations (PDE). Finite difference and finite element methods are perhaps the most widely used numerical solution schemes. Generally, in a numerical method the entire problem is broken up into smaller subregions or elements in which discrete points or values are defined. A system of algebraic functions interconnecting the nodal values at these nodes is derived from the approximation given to the governing equation. How this approximation is made, distinguishes one method from the other.

The finite element (FE) method is a numerical method in which a function is chosen for each subregion to approximate the relation among nodal values defined in each element. These approximation functions are normally polynomials of a lower degree and depend on the geometry of the element and the location of nodal points. The nodal values at these points are the unknowns of the problem. In the finite element method the approximated function in each element is made to satisfy the governing equation in an integral form either by a variational principle or a weighted integral. The substitution of the approximate function intc the integral form of the governing equation for all elements yields a set of equations whose number is equal to that of the unknown nodal values. The solution of these equations for the unknown nodal values represents the approximate solution to the problem. Zienkiewicz [1] showed that the finite element is fairly stable. However, the derivative of the finite element solution may become discontinuous unless further approximation or high degree of the polynomial is used.

In the finite difference (FD) method the functional relationship between a nodal point and its neighboring ones is neither obtained from the analytic solution nor from the approximate functional forms of the differential equation; instead, they are obtained from the different approximation, which essentially is based on the truncated Taylor series expansion of the dependent variables. The finite difference approximation of the differential equations can be written for each unknown nodal value which is interrelated among neighboring nodal values. Thus, unknown nodal values will be governed by the n finite difference equations. This set of the algebraic equations can then be solved as in the case of the finite element method providing the approximate numerical solution. The common difficulty with the finite difference (FD) method, depending on the partial differential equation, is the stability, accuracy, and rate of convergence.

The high-speed computing machine has enabled scientists to solve complex problems. This capability has, in turn, stimulated research in numerical analysis since the effective utilization of computation depends strongly upon the continual advancement of research in relevant areas of mathematical analysis. A good numerical method thus must be able to provide numerical solutions at any point of the problem domain such that the solution is less dependent on the grid size and is accurate with the least truncation errors. Furthermore, the numerical scheme must be stable and have a fast rate of convergence. In the present investigation, a numerical scheme called the finite analytic (FA) method is investigated.

The finite analytic (FA) method is a relatively new numerical method for solving the ordinary and partial differential equations, developed recently by Li and Chen [2] and Chen and Li [3]. The basic idea of the FA method is to incorporate the analytic solution in the numerical solution of partial differential equations. The FA method is neither the finite difference (FD) nor the finite element (FE) method. The FA method utilizes the local analytic solution of the ordinary and partial differential equations obtained for small subregions of the problem. To implement the finite analytic (FA) method, the domain of a complex problem is first subdivided into simple subregions in which the problem may be solved analytically. Secondly, from the local analytic solution an algebraic relation between a nodal value in the subregion and its neighboring nodal values is obtained. If the problem is divided into n subregions there will be n independent algebraic equations to be solved. The solution of the system of finite analytic algebraic equations thus provides the numerical solution of the problem.

Li and Chen [2] and Chen and Li [3] have shown that the FA method has several advantages over the finite difference (FD) and finite element (FE) methods. Firt, the FA method is relatively less dependent on grid size and secondly, the system of FA algebraic equation is relatively stable. Thirdly, the FA solution is differentiable in any direction and is a continuous function in the solution domain. The disadvantage of the FA method is that the method requires analytic analysis.

In the present investigation the FA method is further explored and developed by applying the FA method to solve the Poisson equation. In Chapter II, previous works related to the FA method are reviewed. Since the FA method is relatively new, no previous works done identically in the method resemble the finite analytic method. However, some numerical methods, that are found to be partially similar to the finite analytic method, are mentioned. In Chapter III the principle of the finite analytic (FA) method is outlined. Chapter III describes the basic principle of the finite analytic (FA) method for solving partial differential equations. In Chapter IV the finite analytic (FA) solution is derived for the Poisson equation for different types of subregions. Chapter V illustrates the FA solution of a two-dimensional steady-state heat conduction in a square region with uniform energy generation. In this Chapter the FA solution to the problem is compared with the exact, finite difference (FD), and the finite element (FE) solutions. The application of the FA method in solving the Laplace equation for complex geometry boundaries is given in Chapter VI. This problem can be considered to be the heat conduction problem with irregular solid geometry or the potential flow problem in a contracted channel. The last Chapter presents summaries, conclusions and suggestions.

CHAPTER II
PREVIOUS WORKS

As already mentioned, analytic methods for partial differential equations are usually restricted to very simple geometries and boundary conditions. For the more complex problems, numerical methods must be used to solve the problem.

The finite difference (FD) method which is derived from the truncated Taylor series expansion was used for ordinary differential (ODE) equations by Euler [4] in 1768. For partial differential equations the first computation of the finite difference methods was probably carried out by Rung [5] in 1908 who studied the numerical solution of the Poisson equation. At approximately the same time Richardson [6], in England, was carrying on similar research. In 1918 Liebmann [7], in considering the finite difference approximation to Laplace's equation, suggested an improved method of iteration.

The "best" 9-point finite difference formula was derived by Greenspan [8] which is perhaps one of the most accurate numerical solutions for the Laplace equation. In the 9-point finite difference formula the solution of the center nodal value located (i,j) is made as a function of the immediate surrounding 8 neighboring nodal values located (i,j±1), (i±1,j), (i+1,j±1), and (i-1,j±1). However, the similar derivation has not been carried out for the more complex equations. Also it may not be possible to derive such a similar finite difference formula for the representation of the derivative for the dependent variable at the center node as a function of the neighboring 8 nodal values.

The finite element method is the numerical method based on a variations principle or an integral approximation for a small element of the problem in which the solution is represented by an approximate function, usually a polynomial. The name "finite element" method was first introduced by Clough [9] in 1960, when he solved the two dimensional Poisson equation numerically. Concept of the finite element method was further developed after 1963 when Besseling [10], Melosh [11], Fraejs de Veobeke [12], and Jones [13] recognized that the finite element method was a form of the Ritz method and confirmed it as a general technique to handle elastic continoum problems. In 1965, the finite element method received an even broader interpretation when Zienkiewicz and Chevny [14] reported that it is applicable to all field problems which can be cast into variational form.

The finite analytic method is a numerical scheme based on the analytical solution obtained for a small subregion of the problem in which the governing equation is locally approximated or linearized but retained the differential form. The numerical solution of the problem is then, obtained from the assembly of all analytic solutions. The idea of the finite analytic numerical method including element analytic and line analytic was established by Li and Chen [2] in 1978 and, Chen and Li [3] in 1979. Naseri-Neshat [15] then applied the FA method further to two dimensional Navier-Stokes equation, and demonstrated that the FA mechod is made more stable and accurate for elliptic partial differential equations even at higher Reynolds number.

As it was mentioned the FA method is relatively a new numerical solution scheme that utilizes the local analytic solution of the ordinary or partial differential equation. There are some numerical methods similar to the FA method, but there are some basic differences. Many methods similar to the finite analytic method are based on the concept to reduce the governing partial differential equations to an ordinary differential equation in one direction while the FA method retains the partial differential form. Some of these methods similar to the FA method bear the name of Telenin's method, the method of lines (MOL), and Fourier series (FS) methods. Roach [16] in his book mentioned the Fourier series (FS) methods. The Fourier series methods are based on the fact that an exact solution to the finite difference equation can be expressed in terms of finite eigen function expansion while the FA method does not involve the use of finite difference approximation in its formulation. Basically, the Fourier series methods involved breaking down a complex problem into simpler problems but the simpler problems are approximated by the finite difference operator, which is the source of the truncated error.

CHAPTER III

PRINCIPLES OF THE FINITE ANALYTIC METHOD

In this Chapter the basic idea of the FA method as outlined by Li and Chen [2] is introduced. Consider a partial differential equation $L(\psi) = -\xi$, where L is any partial differential, linear or nonlinear and $\xi(x,y)$ is an inhomogenious term. This partial differential equation (PDE) is to be solved in region D, as shown in Figure (III-1), with the boundary conditions and/or initial conditions to be specified so that the problem is well posed. If the analytic solution to the partial differential equation is available, then there will be no need for the numerical methods. However, in many physical and engineering problems, finding an analytic solution due to either the complexity of the equation or the irregularity of the problem domain is not readily available. Therefore, a numerical method such as the finite analytic (FA) method may be used to obtain a numerical solution.

III.1 The Principle of Finite Analytic Method

The basic idea of the FA method is the incorporation of local analytic solutions in the numerical solutions of partial differential equations (PDE). The total region

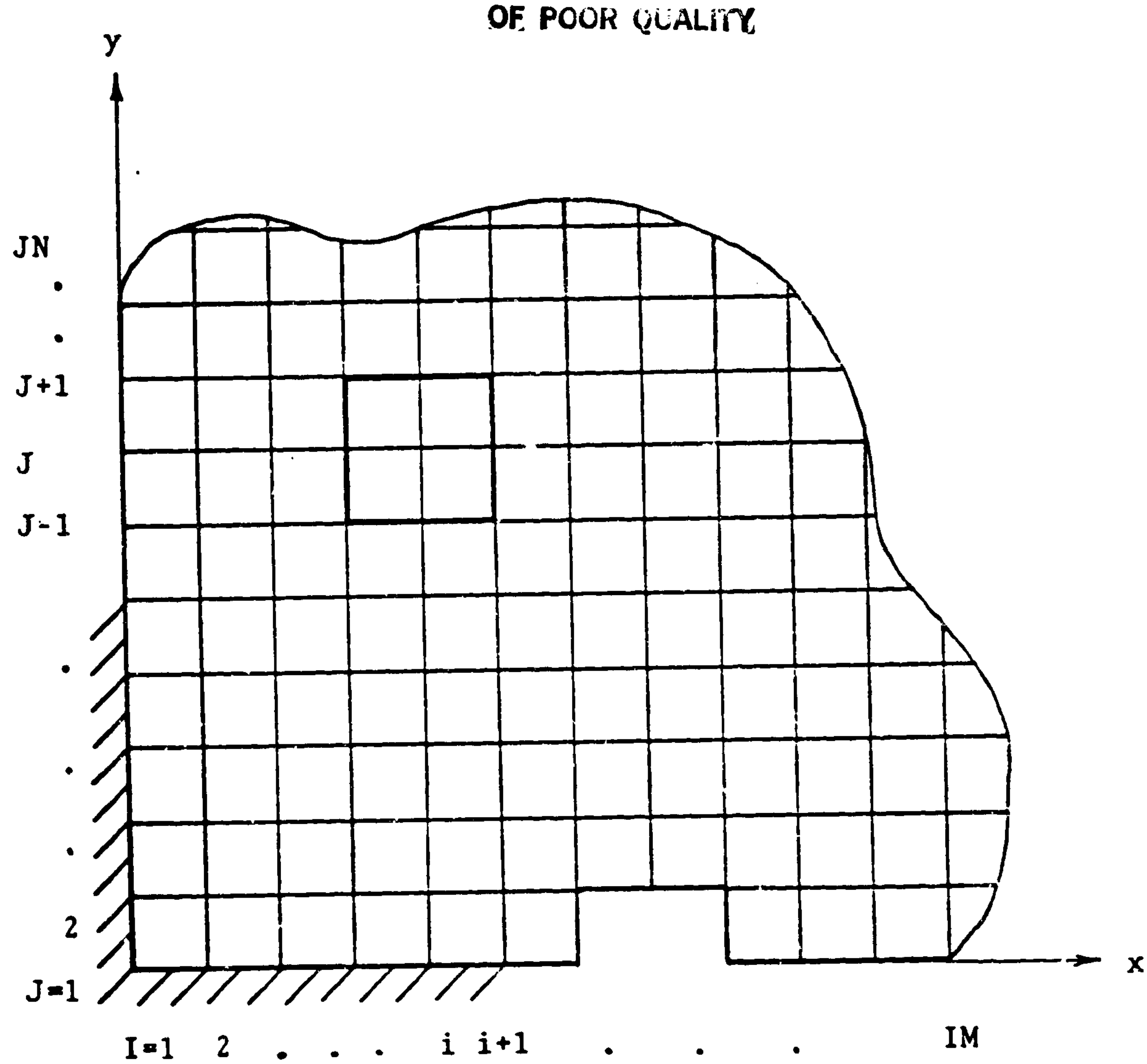

Figure (III-1). Region D

D is decomposed into many small rectangles, as shown in Figure (III-1). The nodal points intersecting the coordinate lines are denoted by, for example, point P, (i,j). A typical subregion of the problem with node points p(i,j) is shown in Figure (III-2), where $h = \Delta x$ and $k = \Delta y$ are grid size in x and y directions. The eight neighboring node points surrounding the point p are denoted by the subscripts EC (east central), WC (west central), SC (south central), NC (north central), NE (northeast), NW (northwest), SE (southeast), and SW (southwest). These points correspond to the points (i+1,j), (i-1,j), (i,j-1), (i,j+1), (i+1,j+1), (i-1,j-1), (i+1,j-1), and (i-1,j-1) respectively.

Once the region D is subdivided into simple rectangular subregions, an analytic solution in a single subregion may still be difficult, such as nonlinear partial differential equations like the Navier-Stokes equation. Since, the subregion is small, the local linearization may be made to obtain an approximate solution. The finite analytic solution of the Navier-Stokes equation was solved by Naseri-Neshat [15]. When the region D has been divided into simple rectangular subregions, the local approximate analytic solution may be found for these simple regions provided the boundary and initial conditions in each simple subregion are properly specified.

In this present investigation only the linear partial differential equations, namely Laplace and Poisson equations with linear boundary conditions,are considered.

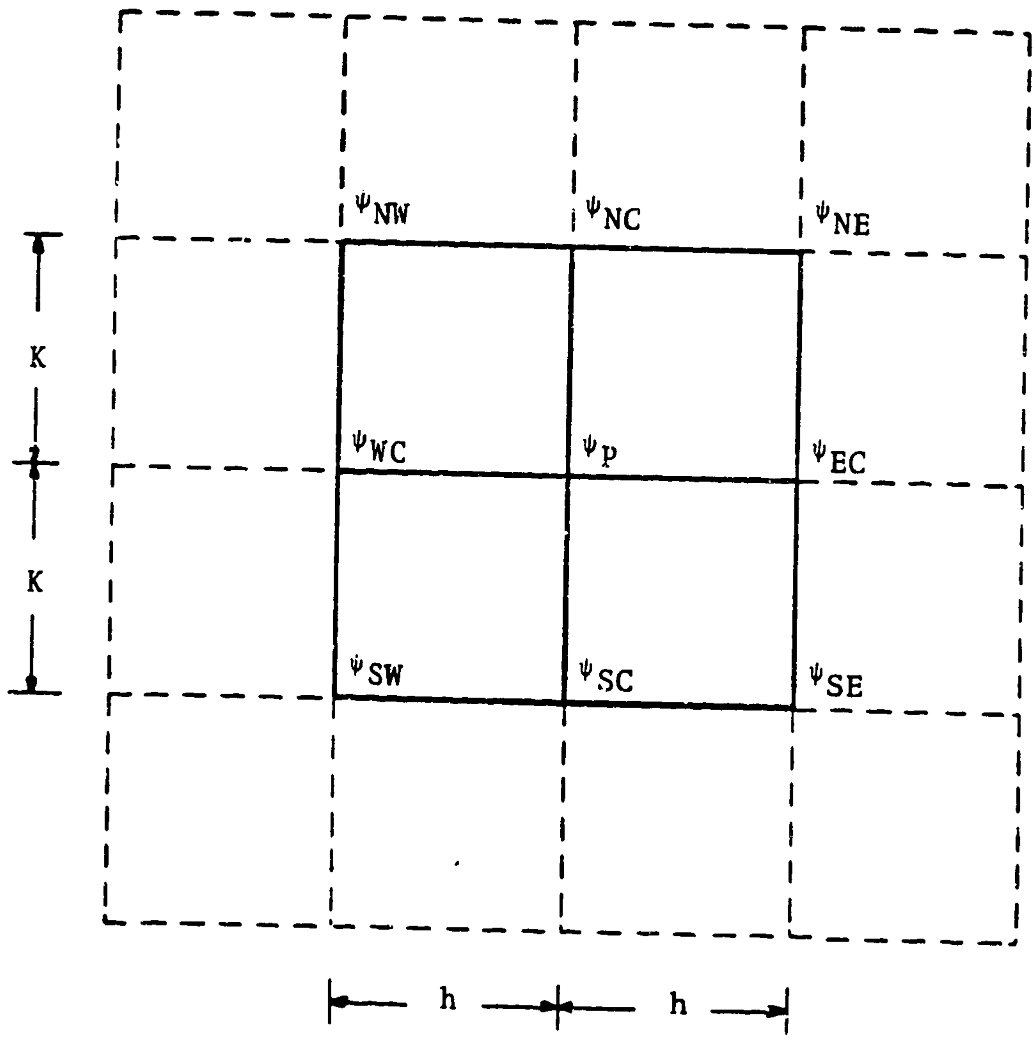

Figure (III-2). A Typical Subregion

III.2 The Finite Analytic Solution to a Subregion

Consider a simple subregion, as shown in Figure (III-2). The elliptic partial equation $L(\psi) = -\xi$ may be solved analytically for the subregion with specified boundary conditions, and the nonhomogeneous term ξ as

$$\psi = f(f_N(x),\ f_S(x),\ f_E(y),\ f_W(y), h, K, x, y, \xi) \tag{III-1}$$

where the f_N, f_S, f_E, and f_W are specified boundary conditions. The north and south boundary conditions f_N, f_S are functions of x while the east and west boundary conditions f_E and f_W are functions of y. For the purpose of the numerical solution,the boundary functions f_N, f_S, f_E, and f_W may be approximately expressed in terms of the nodal values along the boundary such as:

$$\begin{aligned} f_N &= f(\psi_{NW},\ \psi_{NC},\ \psi_{NE},\ h,\ x), \\ f_E &= f(\psi_{NE},\ \psi_{EC},\ \psi_{SE},\ k,\ y), \\ f_S &= f(\psi_{SW},\ \psi_{SC},\ \psi_{SE},\ h,\ x), \\ f_W &= f(\psi_{NW},\ \psi_{NC},\ \psi_{SW},\ k,\ y). \end{aligned} \tag{III-2}$$

The functional relationship between the unknown values of the dependent variable ψ at any interior point (x,y) of the local subregion in terms of its surrounding boundary points ψ_{EC}, ψ_{WC}, ψ_{NC}, ψ_{SC}, ψ_{NE}, ψ_{NW}, ψ_{SE}, and ψ_{SW} can be obtained as,

$$\psi = f(\psi_{EC}, \psi_{WC}, \psi_{NC}, \psi_{SC}, \psi_{NE}, \psi_{NW}, \psi_{SE}, \psi_{SW}, h, k, x, y, \xi) \qquad \text{(III-3)}$$

which is the basic finite analytic formula. For linear operators such as the Laplace operator, the 9-point FA solution for the interior point at P has the form

$$\psi_P = C_{EC}\,\psi_{EC} + C_{WC}\,\psi_{WC} + C_{NC}\,\psi_{NC} + C_{SC}\,\psi_{SC} + C_{NE}\,\psi_{NE} + C_{NW}\,\psi_{NW} + C_{SE}\,\psi_{SE} + C_{SW}\,\psi_{SW} + C_{GP} \qquad \text{(III-4)}$$

where the C's are the finite analytic coefficients whose values are obtained from the local analytic solution. For example, C_{EC} denotes the coefficient multiplying the east center node value ψ_{EC}, C_{GP} is the inhomogeneous part of the local analytic solution. Equation (III-4) is an algebraic equation relating to the interior nodal value ψ_P to its surrounding eight nodal values. It should be noted that equation (III-4) is obtained from the analytic solution rather than from the finite difference or finite element approximation of the partial differential equation.

The same finite analytic procedures may be applied to adjacent subregions where the boundary nodal point, say EC, is considered as the interior point p. Thus, in general, we have n equations similar to equation (III-4) for n unknown nodes (i,j) in the entire region D. They may be written as:

$$\psi_{i,j} = C_{i+1,j}\,\psi_{i+1,j} + C_{i-1,j}\,\psi_{i-1,j} + C_{i,j+1}\,\psi_{i,j+1} + C_{i,j-1}\,\psi_{i,j-1} + C_{i+1,j-1}\,\psi_{i+1,j-1} + C_{i+1,j+1}\,\psi_{i+1,j+1} + C_{i-1,j+1}\,\psi_{i-1,j+1} + C_{i-1,j-1}\,\psi_{i-1,j-1} + C_{i,j}(\xi) \qquad \text{(III-5)}$$

where $i = 1,..IM$, and $j = 1,..JN$. The system of equations given in equation (III-5) is the finite analytic representation of the partial differential equation $L(\psi) = -\xi$. The assembly of all the expressions for all nodal points can then be expressed in a matrix form and can be solved by many existing numerical techniques such as the Gauss-Seidal iterative method or ADI (Alternative Direction Implicit) method.

There is an essential difference between the finite analytic (FA) method just described and the finite difference (FD) or the finite element (FE) methods. In finite difference (FD) methods the relationship between ψ_P and its neighboring points ψ_n is not obtained from the analytic solution of the differential equation, but instead, from the difference formula truncated from the Taylor series expansion of the dependent variable about its neighboring points. On the other hand, the FE method assumes an approximate functional form (shape function), normally some polynomials of a lower degree, say up to the 6th degree, to represent the solution in the whole local element. It uses the variational or the Galerkin type or weighted residual type of integration on the differential equation over the local element to find the relation between ψ_P and its neighboring points ψ_n.

The finite analytic solution given in Eq. (III-3) on the contrary is obtained from the local analytic solution of the differential equation $L(\psi) = -\xi$ without tempering the derivatives of the governing equation. For the Laplace and Poisson equation the only approximation made is on the boundary conditions of the subregions. The accuracy of the finite analytic (FA) solution may be improved by considering more boundary nodal points in the local subregion. For example, the use of five nodal points on each side of the boundary as shown by the dashed line in Figure (III-2), will lead to a more accurate 17-point finite analytic solution.

It should also be noticed here that the finite analytic solution, since it is analytic, is differentiable. Therefore, the derivative of the solution ψ Eqs. (III-3), which represents important physical variables such as heat flux from the temperature distribution or velocity and stress from the potential or stream function, can be readily obtained without the difficulty, or the numerical differentiability is one of the advantages of the finite analytic method. The finite analytic solution for the derivatives may be written as:

$$(\phi_x)_p = C_{xEC}\,\psi_{EC} + C_{xWC}\,\psi_{WC} + C_{xN}\,\psi_{NC} + C_{xSC}\,\psi_{SC} + C_{xNE}\,\psi_{NE} + C_{xNW}\,\psi_{NW} + C_{xSE}\,\psi_{SE} + C_{xSW}\,\psi_{SW} + C_{xGP} \quad \text{(III-6)}$$

and

$$(\psi_y)_p = C_{yEC}\,\psi_{EC} + C_{yWC}\,\psi_{WC} + C_{yNC}\,\psi_{NC} + C_{ySC}\,\psi_{SC} + C_{yNE}\,\psi_{NE} + C_{yNW}\,\psi_{NW} + C_{ySE}\,\psi_{SE} + C_{ySW}\,\psi_{SW} + C_{yGP} \quad \text{(III-7)}$$

where C_x's or C_y's are respectively the finite analytic coefficients multiplying the corresponding neighbor nodal values.

CHAPTER IV

THE FINITE ANALYTIC FORMULA FOR THE POISSON EQUATION

In this Chapter the FA method is applied to the Poisson equation as an example of an elliptic partial differential equation. The Poisson equation, which includes the Laplace equation, appears in many physical as well as engineering problems, such as steady state heat conduction with heat generation or source and sink flows and in fluid dynamics. Several basic FA solutions for the subregion with different local boundary conditions are presented in this chapter. The finite analytic solution for a special problem will be given in the subsequent chapters.

Let us consider an example of a Poisson problem with the boundary condition shown in Figure (IV-1) where the lower left corner is insulated while the outer boundary may be specified with the boundary conditions of Dirichlet (dependent variable), Neumann (derivative) or Churchile (mixed) type. On the other hand, for an interior subregion each boundary condition of the subregion is expressed by three dependent nodal values. The finite analytic solutions then are different if the boundary conditions are different. Therefore, in the following

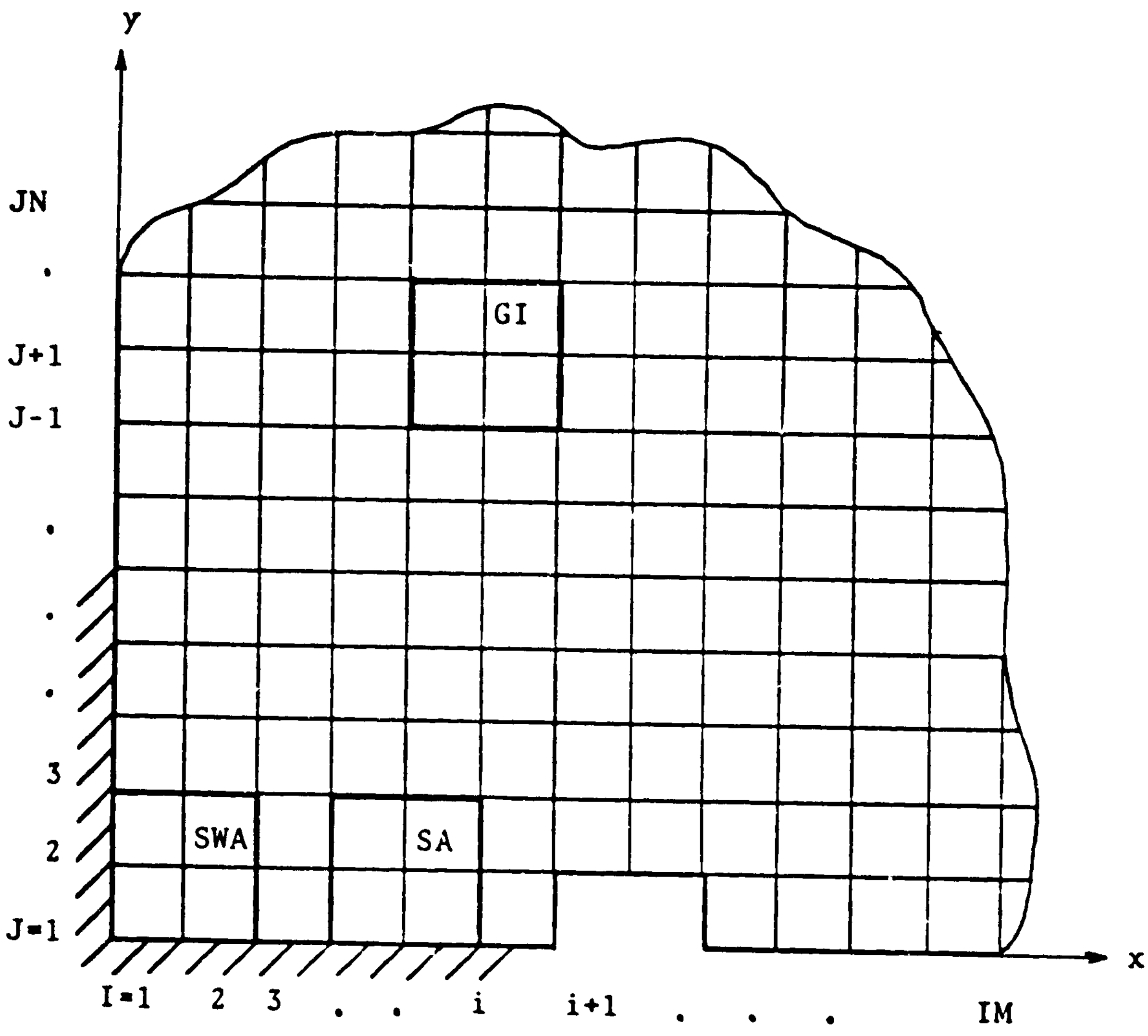

Figure (IV-1). Region D with the Insulated Left Corner section we shall consider the FA solutions for different types of boundary conditions that are present in Figure (IV-1).

IV.1 The FA Solution for General Internal (GI) Subregion

Let us consider now the finite analytic solution for the two-dimensional Poisson equation in the rectangular subregion, shown in Figure (IV-2). This problem represents a typical problem for the internal subregion where only dependent variables are used to specify the boundary condition. The governing equation is

$$\psi_{xx} + \psi_{yy} = -\xi \qquad \text{(IV-1)}$$

where ξ in general can be a function of x and y, but in deriving the FA solution, ξ will be approximated as a constant in the local subregion. The boundary conditions for this subregion are

$$\begin{aligned} x &= 0 & \psi &= f_W(y), \\ x &= 2h & \psi &= f_E(y), \\ y &= 0 & \psi &= f_S(x), \\ y &= 2k & \psi &= f_N(x), \end{aligned} \qquad \text{(IV-2)}$$

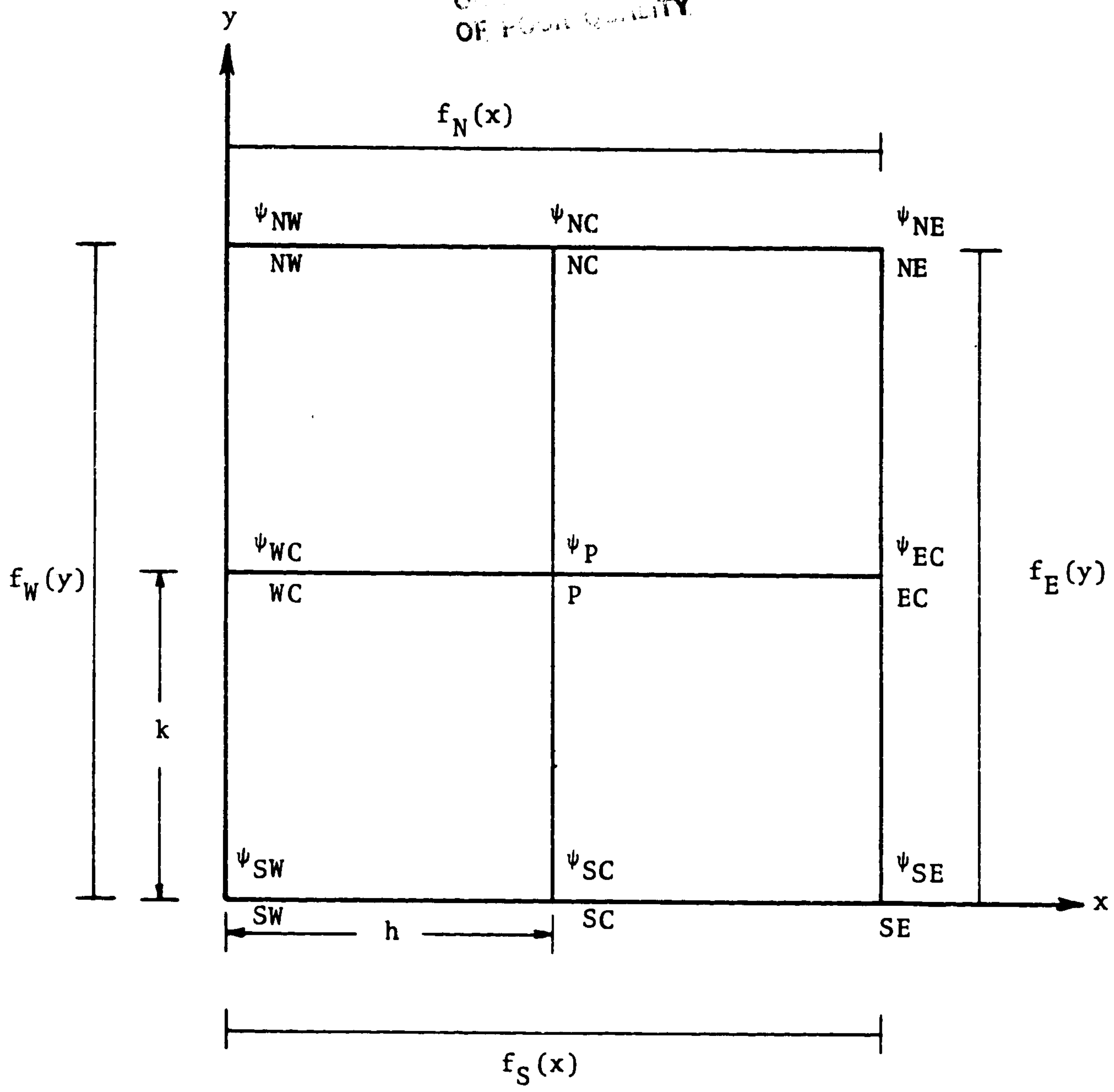

Figure (IV-2). Typical General Internal Subregion where f_W, f_E, f_S, and f_N are west, east, south, and north boundary functions of the subregion. In order to derive a 9-point finite analytic formula, the boundary conditions can be approximately represented by quadratic polynomials in terms of the boundary nodal values at boundary points. For example;

$$f_N(x) = a_0 + a_1 x + a_2 x^2,$$

where

$$a_0 = \psi_{NW},$$

$$a_1 = -\frac{3}{h}\psi_{NW} + \frac{2}{h}\psi_{NC} - \frac{1}{2h}\psi_{NE}, \qquad \text{(IV-3a)}$$

$$a_2 = \frac{1}{2h^2}\psi_{NW} - \frac{1}{h^2}\psi_{NC} + \frac{1}{2h^2}\psi_{NE}$$

Similarly, it can be done for the other three boundary functions $f_S(x)$, $f_W(y)$, and $f_E(y)$.

$$f_S(x) = b_0 + b_1 x + b_2 x^2$$

where

$$b_0 = \psi_{SW},$$

$$b_1 = -\frac{3}{h}\psi_{SW} + \frac{2}{h}\psi_{SC} - \frac{1}{2h}\psi_{SE} \qquad \text{(1V-3b)}$$

$$b_2 = \frac{1}{2h^2}\psi_{SW} - \frac{1}{h^2}\psi_{SC} + \frac{1}{2h^2}\psi_{NE}$$

$$f_W(y) = C_0 + C_1 y + C_2 y^2$$

where

$$C_0 = \psi_{SW},$$

$$C_1 = -\frac{3}{k}\psi_{SW} + \frac{2}{k}\psi_{WC} - \frac{1}{2k}\psi_{NW}, \qquad \text{(IV-3c)}$$

$$C_2 = \frac{1}{2k^2}\psi_{SW} - \frac{1}{k^2}\psi_{WC} + \frac{1}{2k^2}\psi_{NW}$$

and

$$f_E(y) = d_0 + d_1 y + d_2 y^2$$

where

$$d_0 = \psi_{SE},$$

$$d_1 = -\frac{3}{k}\psi_{SE} + \frac{2}{k}\psi_{EC} - \frac{1}{2k}\psi_{NE}, \qquad \text{(IV-3d)}$$

$$d_2 = \frac{1}{2k^2}\psi_{SE} - \frac{1}{k^2}\psi_{EC} + \frac{1}{2k^2}\psi_{NE}$$

Here h and k are the grid size in the x and y directions. The local analytic solution ψ of the Poisson equation may be obtained by the superposition of the following two problems.

Problem (1): Homogeneous equation with nonhomogeneous boundary conditions

$$\nabla^2\psi_H = 0 \qquad \text{(IV-4)}$$

$$\psi_H = f_E(y) \qquad x = 2h$$

$$\psi_H = f_W(y) \qquad x = 0$$

$$\psi_H = f_N(x) \qquad y = 2k$$

$$\psi_H = f_S(x) \qquad y = 0$$

Problem (2): Nonhomogeneous equation with homogeneous boundary conditions

$$\nabla^2 \psi_{NH} = -\xi \qquad \text{(IV-5)}$$

$$\psi_{NH} = 0 \qquad x = 2h$$

$$\psi_{NH} = 0 \qquad x = 0$$

$$\psi_{NH} = 0 \qquad y = 2k$$

$$\psi_{NH} = 0 \qquad y = 0$$

IV.1.1 The Solution to Problem (1)

Since the Poisson equation is linear, the solution to the subregion can be superposed by four simpler solutions, or

$$\psi_H = \psi_{1H} + \psi_{2H} + \psi_{3H} + \psi_{4H} \tag{IV-6}$$

where,

$$(\psi_{1H})_{xx} + (\psi_{1H})_{yy} = 0,$$

$$(\psi_{2H})_{xx} + (\psi_{2H})_{yy} = 0,$$

$$(\psi_{3H})_{xx} + (\psi_{3H})_{yy} = 0,$$

$$(\psi_{4H})_{xx} + (\psi_{4H})_{yy} = 0 \tag{IV-7}$$

with the corresponding boundary conditions

$$y = 0 \qquad \psi_{1H} = 0, \qquad \psi_{2H} = 0, \qquad \psi_{3H} = 0, \qquad \psi_{4H} = f_S(x),$$

$$y = 2k \qquad \psi_{1H} = 0, \qquad \psi_{2H} = 0, \qquad \psi_{3H} = f_N(x), \quad \psi_{4H} = 0,$$

$$x = 0 \qquad \psi_{1H} = 0, \qquad \psi_{2H} = f_W(y), \quad \psi_{3H} = 0, \qquad \psi_{4H} = 0,$$

$$x = 2h \qquad \psi_{1H} = f_E(y), \quad \psi_{2H} = 0, \qquad \psi_{3H} = 0, \qquad \psi_{4H} = 0. \tag{IV-8}$$

The problem for ψ_{1H} may be solved by the method of separation of variables.

$$(\psi_{1H})_{xx} + (\psi_{1H})_{yy} = 0 \tag{IV-9}$$

with boundary conditions

$$y = 0 \qquad \psi_{1H} = 0$$

$$y = 2k \qquad \psi_{1H} = 0$$

$$x = 0 \qquad \psi_{1H} = 0$$

$$x = 2h \qquad \psi_{1H} = f_E(y).$$

The analytic solution for this problem is

$$\psi_{1H} = \sum_{n=1}^{\infty} E_{1n} \operatorname{Sin}(\lambda_n y) \operatorname{Sinh}(\lambda_n x) \tag{IV-10}$$

where,

$\lambda_n = \frac{n\Pi}{2k}$ and the coefficient E_{1n} is determined as

$$E_{1n} = \frac{\int_0^{2k} f_E(y) \sin(\lambda_n y) dy}{\int_0^{2k} \sin^2(\lambda_n y) \sinh(2\lambda_n h) dy} \tag{IV-11}$$

By substituting $f_E(y)$ from the equation (IV-3d) into equation (IV-11) and integrating the equation (IV-11), E_{1n} is expressed in terms of boundary nodal values ψ_{SE}, ψ_{EC}, and ψ_{NE}. Evaluation of ψ_{1H} at point P gives

$$(\psi_{1H})_p = C_{1SE}\, \psi_{SE} + C_{1EC}\, \psi_{EC} + C_{1NE}\, \psi_{NE} \tag{IV-12}$$

where

$$C_{1SE} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b) \frac{\sinh(b\gamma)}{\sinh(2b\gamma)}$$

$$C_{1EC} = \sum_{n=1}^{\infty} \frac{4}{b^3} \sin(b) \frac{\sinh(b\gamma)}{\sinh(2b\gamma)}$$

$$C_{1NE} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b) \frac{\sinh(b\gamma)}{\sinh(2b\gamma)}$$

$b = \frac{n\Pi}{2}$, $\gamma = h/k$.

Similarly, the solution of ψ_{2H}, ψ_{3H}, and ψ_{4H} can be obtained. The solution to problem (1) is then the sum of these solutions, or

$$\psi_H = \psi_{1H} + \psi_{2H} + \psi_{3H} + \psi_{4H}$$

IV.1.2 The Solution to Problem (2)

To solve the problem with the inhomogeneous term one may use the separation of variables method, first to the homogeneous equation with the two x boundary conditions equation (IV-5) to obtain the eigen values. Then, the solution to the problem (2) reduces in finding the function $D_n(y)$ in the series solution assumed for equation (IV-5) as

$$\psi_{NH}\ (x,y) = \sum_{n=1}^{\infty} D_n(y)\ \sin(\mu_n x) \qquad \text{(IV-13)}$$

where $\mu_n = \frac{n\Pi}{2h}$; $n = 1,2,\ldots$

The unknown function $D_n(y)$ in equation (IV-13) is governed by Eq. (IV-14) which is obtained when Eq. (IV-13) is substituted into the equation (IV-5).

$$\sum_{n=1}^{\infty} [D_n(y) - \mu^2_n\ D_n(y)]\ \sin(\mu_n x) = -\ \xi \qquad \text{(IV-14)}$$

The constant ξ can be expanded in terms of Fourier sine series with the eigen function given in Eq. (IV-13) or

$$\xi = \sum_{n=1}^{\infty} W_n(y) \sin (\mu_n x) \tag{IV-15}$$

where $W_n(y)$ can be found as

$$W_n(y) = \frac{1}{h} \int_{-h}^{h} \xi \sin(\mu_n x) dx$$

or

$$W_n(y) = - \frac{1}{b} \cos(2b) \tag{IV-16}$$

where $b = \frac{n\Pi}{2}$.

Then the Fourier series expansion for ξ is

$$\xi = \sum_{n=1}^{\infty} [- \frac{1}{b} \cos(2b)] \sin(\mu_n x) \tag{IV-17}$$

Substituting equation (IV-17) for ξ into equation (IV-14) leads to a second order ordinary differential equation for $D_n(y)$. The second order differential equation for $D_n(y)$ with its two zero boundary conditions is

$$\begin{aligned} D''_n(y) - \mu^2{}_n D_n(y) &= - W_n(y) \\ D_n(y) = 0 \qquad & y = 0 \\ D_n(y) = 0 \qquad & y = 2K \end{aligned} \tag{IV-18}$$

which has the solution

$$\psi_{NH} = \sum_{n=1}^{\infty} \frac{2h^2 \xi}{b^3} \left[\left(- \frac{1}{\sinh(2b/\gamma)} + \frac{1}{\tanh(2b/\gamma)} \right) \right.$$

$$\left. \sinh(\mu_n y) - \cosh(\mu_n y) + 1 \right] \cos(2b) \sin(\mu_n x) \qquad \text{(IV-19)}$$

At point p where x = h, y = k, the solution becomes

$$(\psi_{NH})_p = \sum_{n=1}^{\infty} \frac{2h^2 \xi}{b^3} \sin(b) \sinh(b/\gamma) \left[\frac{1}{\tanh(2b/\gamma)} - \right.$$

$$\left. \frac{1}{\sinh(2b/\gamma)} + \frac{1}{\sinh(b/\gamma)} - \frac{1}{\tanh(b/\gamma)} \right] \qquad \text{(IV-20)}$$

where $b = n\Pi/2$, $n = 1,2,\ldots$

IV.1.3 The 9-Point FA Formula for the Internal Subregion

The local analytic solution to the subregion shown in Figure (IV-2) is the sum of the homogeneous and nonhomogeneous solutions just obtained in the above section. That is,

$$\psi(x,y) = \sum_{n=1}^{\infty} E_{1n} \sin(\lambda_n y) \sinh(\lambda_n x) + \sum_{n=1}^{\infty} E_{2n} \sin(\lambda_n y)$$

$$[\sinh(\lambda_n y) - \tanh(2\lambda_n h) \cosh(\lambda_n x)] + \sum_{n=1}^{\infty} E_{3n}$$

$$\sin(\mu_n x)\ \sinh(\mu_n y) + \sum_{n=1}^{\infty} E_{4n} \sin(\mu_n x)\ [\sinh(\mu_n y)$$

$$-\tan(2\mu_n k)\ \cosh(\mu_n y)] + \sum_{n=1}^{\infty} \frac{2h^2\xi}{b^3} [(-\frac{1}{\sinh(2b/\gamma)} +$$

$$\frac{1}{\tanh(2b/\gamma)})\ \sinh(\mu_n y) - \cosh(\mu_n y) + 1]\ \cos(2b)$$

$$\sin(\mu_n x) \tag{IV-21}$$

where E_{1n} is equation (IV-12) and

$$E_{2n} = -\frac{\int_o^{2k} f_w(y)\ \sin(\lambda_n y)dy}{K\ \tanh(2\lambda_n h)},$$

$$E_{3n} = \frac{\int_o^{2h} f_N(x)\ \sin(\mu_n x)dx}{h\ \sinh(2\mu_n k)},$$

$$E_{4n} = -\frac{\int_o^{2h} f_s(x)\ \sin(\mu_n x)dx}{h\ \tanh(2\mu_n k)} \tag{IV-22}$$

$$b = \frac{n\Pi}{2},\ \mu_n = \frac{n\Pi}{2h},\ \lambda_n = \frac{n\Pi}{2k}.$$

Substituting the approximate quadratic expressions for the boundary function f's equations (IV-3) into the equation (IV-21) and evaluating it at P, gives the 9-point FA formula as follows:

$$\psi_p = C_{EC}\,\psi_{EC} + C_{WC}\,\psi_{WC} + C_{NC}\,\psi_{NC} + C_{SC}\,\psi_{SC} +$$

$$C_{NE}\,\psi_{NE} + C_{SE}\,\psi_{SE} + C_{NW}\,\psi_{NW} + C_{SW}\,\psi_{SW} + C_{GP} \tag{IV-23}$$

where

$$C_{EC} = \sum_{n=1}^{\infty} \frac{4}{b^3} \sin(b) \frac{\sinh(b\gamma)}{\sinh(2b\gamma)}$$

$$C_{WC} = \sum_{n=1}^{\infty} \frac{4}{b^3} \sin(b) \frac{\sinh(b\gamma)}{\sinh(2b\gamma)}$$

$$C_{NC} = \sum_{n=1}^{\infty} \frac{4}{b^3} \sin(b) \frac{\sinh(b/\gamma)}{\sinh(2b/\gamma)}$$

$$C_{SC} = \sum_{n=1}^{\infty} \frac{4}{b^3} \sin(b) \frac{\sinh(b/\gamma)}{\sinh(2b/\gamma)}$$

$$C_{NE} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b) \left[\frac{\sinh(b\gamma)}{\sinh(2b\gamma)} + \frac{\sinh(b/\gamma)}{\sinh(2b/\gamma)}\right]$$

$$C_{NW} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b) \left[\frac{\sinh(b\gamma)}{\sinh(2b\gamma)} + \frac{\sinh(b/\gamma)}{\sinh(2b/\gamma)}\right]$$

$$C_{SE} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b) \left[\frac{\sinh(b\gamma)}{\sinh(2b\gamma)} + \frac{\sinh(b/\gamma)}{\sinh(2b/\gamma)}\right]$$

$$C_{SW} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b) \left[\frac{\sinh(b\gamma)}{\sinh(2b\gamma)} + \frac{\sinh(b/\gamma)}{\sinh(2b/\gamma)}\right]$$

$$C_{GP} = \sum_{n=1}^{\infty} \frac{2h^2\xi}{b^3} \sin(b) \sinh(b/\gamma) \left[\frac{1}{\tanh(2b/\gamma)} - \right.$$

$$\left. \frac{1}{\sinh(2b/\gamma)} + \frac{1}{\sinh(b/\gamma)} - \frac{1}{\tanh(b/\gamma)}\right]$$

For the Laplace equation since $\xi = 0$, $C_{GP} = 0$.
The coefficient C's in the above equations need to be evaluated once if the grid size is given. In particular, when the grid sizes in x and y directions are the same, that is $\gamma = 1$, the 9-point FA formula for the interior point P, ψ_p, in the interior subregion can be evaluated once. The numerical values of these coefficients are listed in Table (IV-1). Here the numbers given in Table (IV-1) are corresponding values at any given node. For example, 0.205315 is the FA coefficient C_{NC}, C_{SC}, C_{ES}, and C_{WC}. It should be mentioned that the eight coefficients for the neighboring nodal values denoted by ψ_n(i.e. n = EC, WC, etc.) and the constant multiplying ξh^2 are universal for all subregions and also independent of grid size. However, the accuracy of the formula is restricted by the accuracy of the approximation made for the boundary condition in equation (IV-3) which has an accuracy of $O(h^2)$ or $O(k^2)$.

As it was already mentioned that the FA method also gives the solutions to the derivatives $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$. This is a definite advantage for the method as the analytic solution in any subregion is differentiable. Therefore, by differentiating of the equation (IV-21), a corresponding 9-point FA formula for the derivative of a node point inside the rectangular subregion can be derived. For example, the differentiation of equation (IV-21) with respect to x and

$\psi_p =$

NW 0.044685	NC 0.205315	NE 0.044685
WC 0.205315	P	EC 0.205315
SW 0.044685	SC 0.205315	SE 0.044685

$\times\ \psi_n +$

$\xi h^2 (0.29493)$

Table (IV-1). FA Numerical Values for Coefficients in a General Internal Subregion. (The 9-point FA Formula for General Subregions) evaluated at the east-center point along the east boundary, gives the expression for the derivatives at the east central (EC) node of the rectangle, or $\frac{\partial\psi}{\partial x}|_{EC}$ as

$$\frac{\partial\psi}{\partial x}\Big|_{EC} = \frac{1}{h}\,[C_{xEC}\,\psi_{EC} + C_{xWC}\,\psi_{WC} + C_{xNC}\,\psi_{NC} + C_{xSC}\,\psi_{SC} + C_{xNE}\,\psi_{NE} + C_{xSE}\,\psi_{SE} + C_{xNW}\,\psi_{NW} + C_{xSW}\,\psi_{SW}] + C_{xG} \qquad \text{(IV-24)}$$

where

$$C_{xEC} = \sum_{n=1}^{\infty} \frac{4}{b^2}\,\frac{\sin(b)}{\tanh(2b\gamma)}$$

$$C_{xWC} = \sum_{n=1}^{\infty} \frac{4}{b^2}\gamma\,\frac{\sin(b)}{\tanh(2b\gamma)}$$

$$C_{xNC} = \sum_{n=1}^{\infty} \frac{4}{b^2}\gamma\,\frac{\sin(b)}{\tanh(2b\gamma)}$$

$$C_{xSC} = \sum_{n=1}^{\infty} \frac{4}{b^2}\gamma\,\frac{\sin(b)}{\tanh(2b\gamma)}$$

$$C_{xNE} = \sum_{n=1}^{\infty} (1 - \frac{2}{b^2})\gamma\,\sin(b)\,[\frac{1}{\tanh(2b\gamma)} + \frac{1}{\tanh(2b/\gamma)}]$$

$$C_{xSE} = \sum_{n=1}^{\infty} (1 - \frac{2}{b^2})\gamma\,\sin(b)\,[\frac{1}{\tanh(2b\gamma)} + \frac{1}{\tanh(2b/\gamma)}]$$

$$C_{xNW} = \sum_{n=1}^{\infty} (1 - \frac{2}{b^2})\gamma \sin(b) [\frac{1}{\tanh(2b\gamma)} + \frac{1}{\tanh(2b/\gamma)}]$$

$$C_{xSW} = \sum_{n=1}^{\infty} (1 - \frac{2}{b^2})\gamma \sin(b) [\frac{1}{\tanh(2b\gamma)} + \frac{1}{\tan(2b/\gamma)}]$$

$$C_{xG} = \sum_{n=1}^{\infty} \frac{2h\xi}{b^2} [\frac{1}{\tanh(2b/\gamma)} - \frac{1}{\sinh(2b/\gamma)} + \frac{1}{\sinh(b/\gamma)} - \frac{1}{\tanh(b/\gamma)}] \sinh(b/\gamma)$$

For $\gamma = 1$ the 9-point FA formula of Poisson equation for the derivative $\frac{\partial\psi}{\partial x}$ evaluated at the east-center node is written in Table (IV-2). Again, the numerical values in the Table (IV-2) are universal and independent of grid size.

IV.2 The FA Solution for a Subregion with One Side Insulated

The need for deriving the local finite analytic solution for different boundary conditions other than the one just considered in the previous section arises when the subregion has a boundary of the original problem. As shown in Figure (IV-1) different boundary conditions such as derivative of temperature or symmetry may be presented. In this case the finite analytic solution for the subregion is different from the previous one given in section IV-1. In this section the Poisson equation Eq. (IV-1) in the subregion

$$\left.\frac{\partial\psi}{\partial x}\right|_{EC} = \frac{1}{h}\begin{array}{|c|c|c|}\hline \text{NW} & \text{NC} & \text{NE} \\ -0.017485 & -0.324686 & -0.333144 \\ \hline \text{WC} & & \text{EC} \\ -0.140345 & & 1.490972 \\ \hline \text{SW} & \text{SC} & \text{SE} \\ -0.017485 & -0.324686 & -0.333144 \\ \hline \end{array} \times \psi_n + h\xi(0.950832)$$

Table (IV-2). The 9-Point FA Formula of the Poisson Equation for the Derivative $\partial\psi/\partial x$ Evaluated at the Midpoint (EC) of the Boundary Side.

SA with insulated boundary as shown in Figure (IV-3) is solved. The boundary conditions for this subregion are:

$$y = 0 \quad \frac{\partial\psi}{\partial y} = 0$$

$$y = 2k \quad \psi = f_N(x)$$

$$x = 0 \quad \psi = f_W(y)$$

$$x = 2h \quad \psi = f_E(y) \tag{IV-25}$$

Again, since the Poisson equation is linear the solution to the problem, ψ, for subregion SA, may be obtained by superposition of the following solutions ψ_N and ψ_{NH} to the two simpler problems.

Problem (1): Homogeneous equation with nonhomogeneous boundary conditions

$$\nabla^2 \psi_H = 0$$

$$\psi_H = f_E(y) \qquad x = 2h$$

$$\psi_H = f_W(y) \qquad x = 0 \tag{IV-26}$$

$$\psi_H = f_N(x) \qquad y = 2k$$

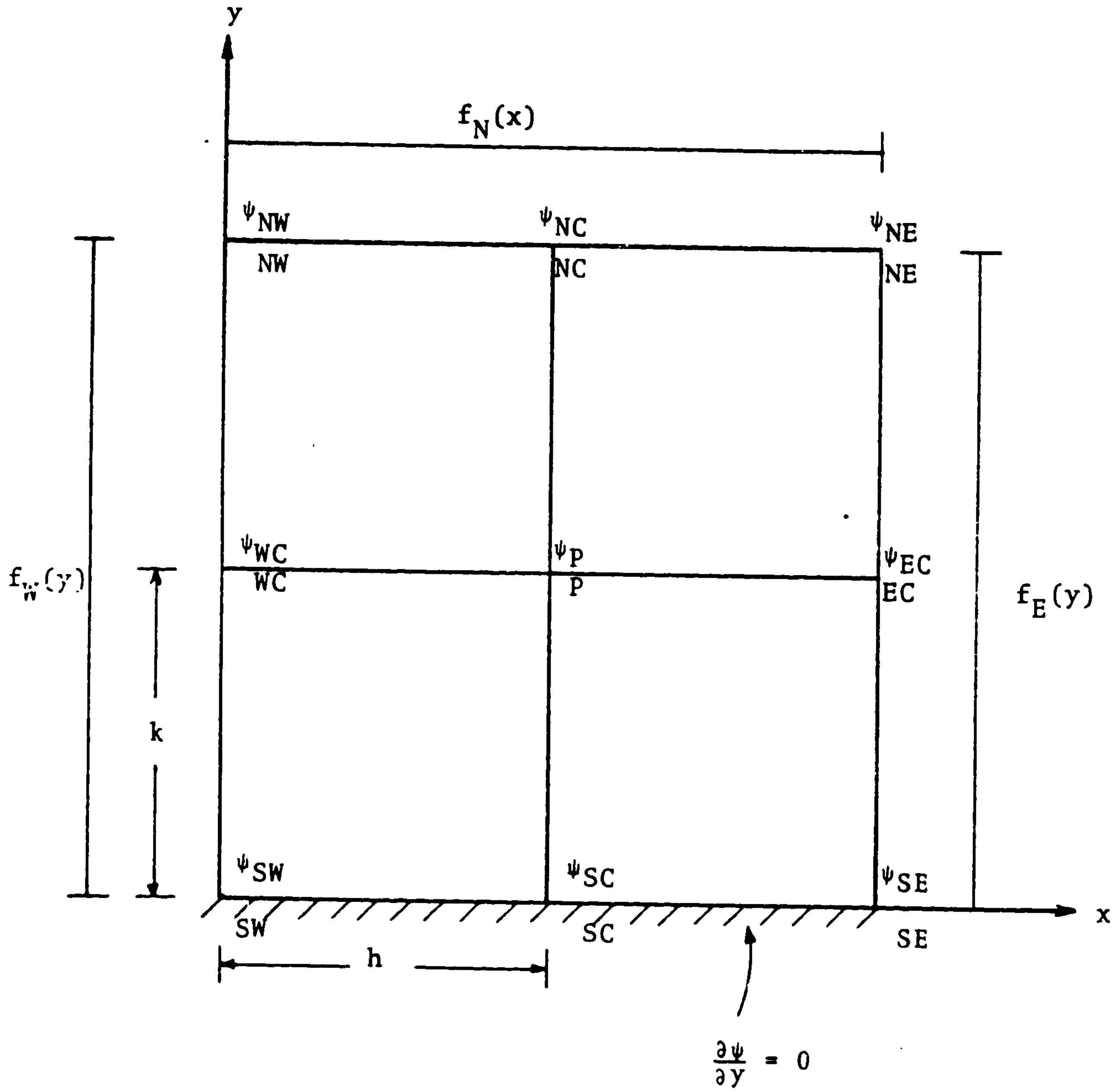

Figure (IV-3). Subregion SA (South Adiabatic)

$$\partial \psi_H/\partial y = 0 \qquad y = 0 \tag{IV-26}$$

The homogeneous equation with nonhomogeneous boundary conditions can be divided to three simpler problems, each having three homogeneous and one non-zero boundary conditions as,

$$\psi_H = \psi_{1H} + \psi_{2H} + \psi_{3H}$$

where

$$\nabla^2 \psi_{1H} = 0$$

$$\psi_{1H} = f_N(x) \qquad y = 2k$$

$$\partial \psi_{1H}/\partial y = 0 \qquad y = 0$$

$$\psi_{1H} = 0 \qquad x = 2h$$

$$\psi_{1H} = 0 \qquad x = 0 \tag{IV-27a}$$

$$\nabla^2 \psi_{2H} = 0$$

$$\psi_{2H} = 0 \qquad y = 2k$$

$$\partial \psi_{2H}/\partial y = 0 \qquad y = 0 \tag{IV-27b}$$

$$\psi_{2H} = 0 \qquad x = 2h$$

$$\psi_{2H} = f_W(y) \quad x = 0 \tag{IV-27b}$$

$$\nabla^2 \psi_{3H} = 0$$

$$\psi_{3H} = f_N(x) \quad y = 2k$$

$$\partial\psi_{3H}/\partial y = 0 \quad y = 0 \tag{IV-27c}$$

$$\psi_{3H} = 0 \qquad x = 2h$$

$$\psi_{3H} = 0 \qquad x = 0$$

The solution to ψ_{1H}, ψ_{2H} and ψ_{3H} can be carried out by the method of separation of variables and added together to give the solution to problem (1). The solution to the equation (IV-27a) has the form

$$\psi_{1H} = \sum_{n=1}^{\infty} D_n \ \sin(\lambda_n x) \ \cosh(\lambda_n y) \tag{IV-28}$$

where $\lambda_n = \frac{n\pi}{2h}$ is the eigen values and $n = 1,2,3....$

The coefficient D_n is determined as

$$D_n = \frac{\int_0^{2h} f_N(x) \ \sin(\lambda_n x)dx}{\int_0^{2h} \sin^2(\lambda_n x) \ \cosh(2\lambda_n k)dx}$$

Substituting $f_N(x)$ from equation (IV-3a) into equation (IV-28) and evaluating ψ_{1H} at point p gives,

$$(\psi_{1H})_p = C_{1NW}\,\psi_{NW} + C_{1NC}\,\psi_{NC} + C_{1NE}\,\psi_{NE} \tag{IV-29}$$

where

$$C_{1NW} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b)\, \frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}$$

$$C_{1NC} = \sum_{n=1}^{\infty} \frac{4}{b^3} \sin(b)\, \frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}$$

$$C_{1NE} = \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \sin(b)\, \frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}$$

Similarly, the solution to $(\psi_{2H})_p$ and $(\psi_{3H})_p$ can be obtained by rotating the coordinates accordingly. Then

$$(\psi_H)_p = (\psi_{1H})_p + (\psi_{2H})_p + (\psi_{3H})_p \tag{IV-30}$$

Problem (2): Nonhomogeneous equation with homogeneous boundary conditions

$$\begin{aligned} \nabla^2 \psi_{NH} &= -\xi \\ \psi_{NH} &= 0 \qquad x = 2h \end{aligned} \tag{IV-31}$$

$$\psi_{NH} = 0 \qquad x = 0$$

$$\psi_{NH} = 0 \qquad y = 2k \tag{IV-31}$$

$$\partial \psi_{NH}/\partial y = 0 \qquad y = 0$$

As it was done in section IV.1.2 for equation (IV-5) the problem (IV-31) may be solved. Evaluating the ψ_{NH} at point p one has

$$(\psi_{NH})_p = \sum_{n=1}^{\infty} \frac{2h^2 \xi}{b^3} \left[1 - \frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}\right] \sin(b) \tag{IV-32}$$

where $b = \frac{n\Pi}{2}$ and $\gamma = h/k$.

The 9-point finite analytic (FA) formula for the sub-region SA, is found by superposing the solution ψ_H and ψ_{NH} given equations (IV-30) and (IV-32).

$$\psi_p = (\psi_H)_p + (\psi_{NH})_p$$

or

$$\psi_p = C_{EC}\,\psi_{EC} + C_{WC}\,\psi_{WC} + C_{NC}\,\psi_{NC} + C_{SC}\,\psi_{SC} + C_{NE}\,\psi_{NE}$$

$$+ C_{NW}\,\psi_{NW} + C_{SE}\,\psi_{SE} + C_{SW}\,\psi_{SW} + C_{GP} \tag{IV-33}$$

where

$$C_{EC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a)\right] \cos\left(\frac{a}{2}\right) \frac{\sinh(a\gamma/2)}{\sinh(a\gamma)}$$

$$C_{WC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a)\right]\left[\cosh\left(\frac{a\gamma}{2}\right) - \frac{\sinh(a\gamma/2)}{\tanh(a\gamma)}\right]$$

$$\cos\left(\frac{a}{2}\right)$$

$$C_{NC} = \sum_{n=1}^{\infty} \frac{4}{b^3}\sin(b)\,\frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}$$

$$C_{SC} = 0$$

$$C_{NE} = \sum_{n=1}^{\infty} \left[\frac{2}{a^2} + \left(\frac{2}{a} - \frac{8}{a^3}\right)\sin(a)\right] \frac{\sinh(a\gamma/2)}{\sinh(a\gamma)}$$

$$\cos(a/2) + \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right)\sin(b)\,\frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}$$

$$C_{NW} = \sum_{n=1}^{\infty} \left[\frac{2}{a^2} + \left(\frac{2}{a} - \frac{8}{a^3}\right)\sin(a)\right]\left[\cosh(a\gamma/2) - \frac{\sinh(a\gamma/2)}{\tanh(a\gamma)}\right]$$

$$\cos(a/2) + \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right)\sin(b)\,\frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}$$

$$C_{SE} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3}\sin(a)\right] \frac{\sinh(a\gamma/2)}{\sinh(a\gamma)} \cos(a/2)$$

$$C_{SW} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3} \sin(a)\right]\left[\cosh(a\gamma/2) - \frac{\sinh(a\gamma/2)}{\tanh(a\gamma)}\right]$$

$$\cos(a/2)$$

$$C_{GP} = \sum_{n=1}^{\infty} \frac{2h^2}{b^3} \left[1 - \frac{\cosh(b/\gamma)}{\cosh(2b/\gamma)}\right] \sin(b),$$

$$a = \frac{(2n-1)\Pi}{2}, \quad b = \frac{n\Pi}{2}, \quad \gamma = \frac{h}{k}, \text{ and } n = 1,2....$$

For a given ratio of grid size say $\gamma = 1$, the coefficient C's in the above equations can be evaluated.

The 9-point FA formula for the subregion of the type SA is in Table (IV-3) where the numerical values for the coefficients are again universal and independent of the grid size. The hash marks drawn at the bottom of the Table denote the south boundary's insulation or $\frac{\partial\psi}{\partial y} = 0$.

IV.3 The FA Solution for a Subregion with Two Insulated Sides

Consider the subregion SWA, shown in Figure (IV-4) where the south and west boundaries are insulated or have zero derivatives. The governing equation and boundary conditions are

ψ_p =

NW 0.042678	NC 0.223055	NE 0.042678
WC 0.271649	P	EC 0.271649
SW 0.741445	SC 0.0	EC 0.741445

x ψ_n + ξh^2(0.338716)

Table (IV-3). FA Numerical Values for Coefficients in a Subregion with One Insulate Boundary. (The 9-Point FA Formula for Subregion SA)

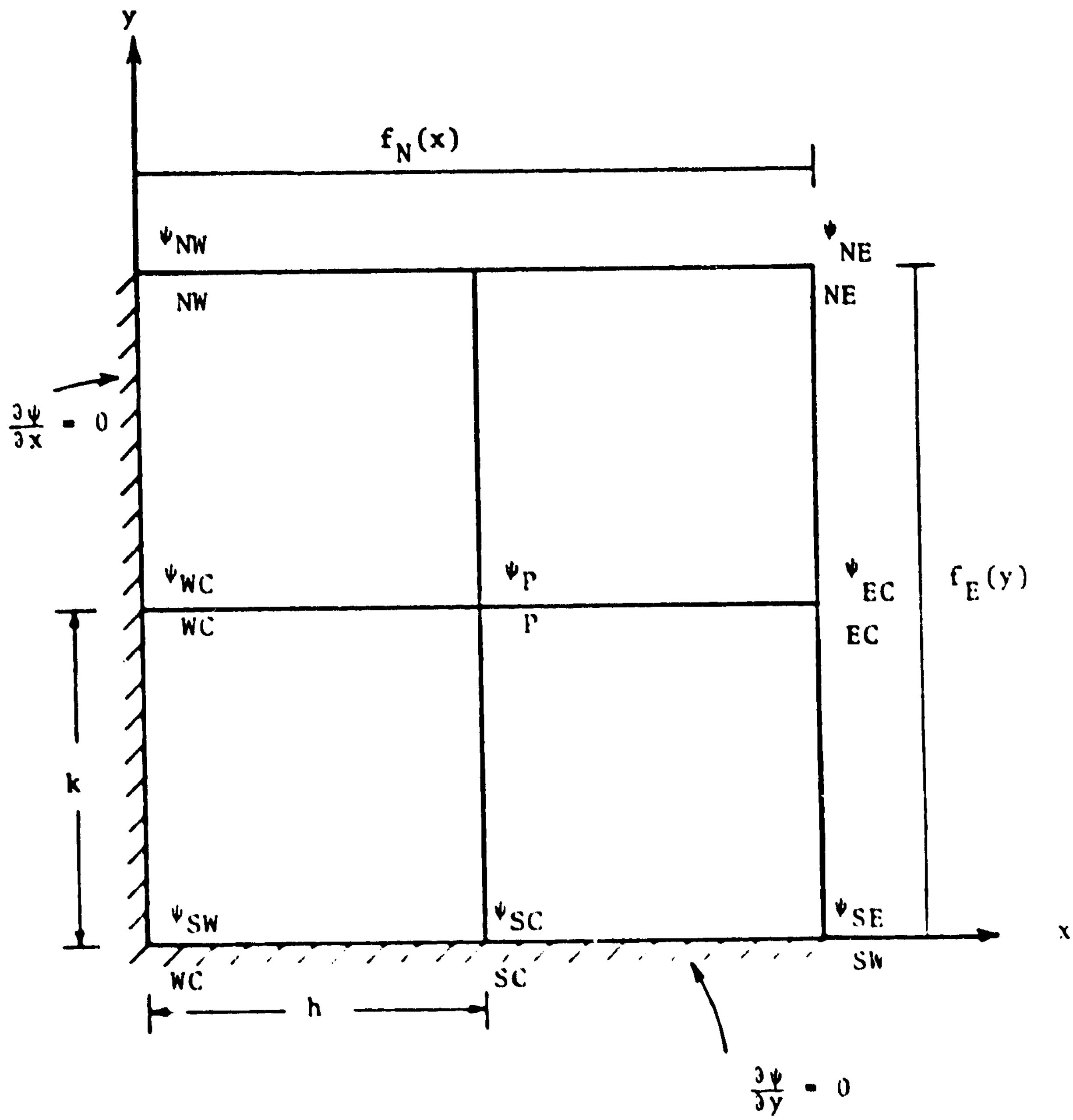

Figure (IV-4). Subregion SWA

$$\nabla^2 \psi = -\xi$$

$$y = 0 \qquad \frac{\partial \psi}{\partial y} = 0$$

$$y = 2k \qquad \psi = f_N(x)$$

$$x = 0 \qquad \frac{\partial \psi}{\partial x} = 0$$

$$x = 2h \qquad \psi = f_E(y) \tag{IV-34}$$

The solution to equation (IV-34) can be similarly solved by the method of separation of variables. Thus, the finite analytic solution for subregion SWA evaluated at point P gives

$$\psi_p = C_{EC}\,\psi_{EC} + C_{WC}\,\psi_{WC} + C_{NC}\,\psi_{NC} + C_{SC}\,\psi_{SC} + C_{NE}\,\psi_{NE}$$

$$+ C_{NW}\,\psi_{NW} + C_{SE}\,\psi_{SE} + C_{SW}\,\psi_{SW} + C_{GP} \tag{IV-35}$$

where

$$C_{EC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a) \right] \frac{\cosh(a\gamma/2)}{\cosh(a\gamma)} \cos(a/2)$$

$$C_{WC} = 0$$

$$C_{NC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a)\right] \frac{\cosh(a/2\gamma)}{\cosh(a/\gamma)} \cos(a/2)$$

$$C_{SC} = 0$$

$$C_{NE} = \sum_{n=1}^{\infty} \left[\frac{2}{a^2} + \left(\frac{2}{a} - \frac{8}{a^3}\right)\sin(a)\right]\left[\frac{\cosh(a\gamma/2)}{\cosh(a\gamma)} + \frac{\cosh(a/2\gamma)}{\cosh(a/\gamma)}\right] \cos(a/2)$$

$$C_{NW} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3}\sin(a)\right] \frac{\cosh(a/2\gamma)}{\cosh(a/\gamma)} \cos(a/2)$$

$$C_{SE} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3}\sin(a)\right] \frac{\cosh(a\gamma/2)}{\cosh(a\gamma)} \cos a/2$$

$$C_{SW} = 0$$

$$C_{GP} = \sum_{n=1}^{\infty} \frac{8h^2\xi}{a^3}\sin(a)\left[1 - \frac{\cosh(a/2\gamma)}{\cosh(a/\gamma)}\right] \cos(a/2)$$

$$a = \frac{(2n-1)\pi}{2},\ n = 1,2,\ldots.$$

Again in the case $\gamma = 1$, the coefficients C's are universal constants. Once these coefficients are calculated, they can be saved and used repeatedly. The numerical values of these coefficients are listed in Table (IV-4).

Presented above are some sample FA solutions for the Poisson equation. Further FA solutions can be solved for more complex boundary conditions as the need arises.

$\psi_p =$

NW 0.112834	NC 0.366502	NE 0.0413275
WC 0.0	P	EC 0.366502
SW 0.0	SC 0.0	SE 0.112834

$x\psi_n +$

$\xi h^2(0.724753)$

Table (IV-4). FA Numerical Values for Coefficients in a Subregion with Two Insulated Boundaries. (The 9-Point FA Formula for Subregion SWA)

CHAPTER V

TWO DIMENSIONAL HEAT CONDUCTION WITH CONSTANT HEAT GENERATION

In order to illustrate and examine the accuracy of the finite analytic (FA) method, a problem having an exact solution is solved with the FA method. The finite analytic solution is then compared with the exact solution and the corresponding finite difference and finite element solutions. In this section, the problem of a two dimensional heat conduction in a square region (LxL) with uniform heat generation,as described in Figure (V-1),is chosen. The western and southern sides are insulated while the other two sides are kept at a constant temperature T_L. The governing equation and boundary conditions of the problem are

$$K(T_{xx} + T_{yy}) + g = 0 \tag{V-1}$$

$$x = 0, \qquad y > 0, \qquad \frac{\partial T}{\partial x} = 0$$

$$x = L, \qquad y > 0, \qquad T = T_L$$

$$y = 0, \qquad x > 0, \qquad \frac{\partial T}{\partial y} = 0$$

$$y = L, \qquad x > 0, \qquad T = T_L \tag{V-2}$$

where, g is the heat generation, k the thermal conductivity, and L the size of square.

The problem may be normalized as

$$\psi_{xx} + \psi_{yy} = -1 \tag{V-3}$$

with dimensionless variables defined as

$$\psi = \frac{T - T_L}{gL^2}, \qquad x = \frac{x}{L}, \text{ and } y = \frac{y}{L}.$$

The corresponding boundary conditions in dimensionless forms are

$$\psi_x = 0 \qquad x = 0 \qquad y > 0$$

$$\psi_y = 0 \qquad y = 0 \qquad x > 0$$

$$\psi = 0 \qquad x = 1 \qquad y > 0$$

$$\psi = 0 \qquad y = 1 \qquad x > 0 \tag{V-4}$$

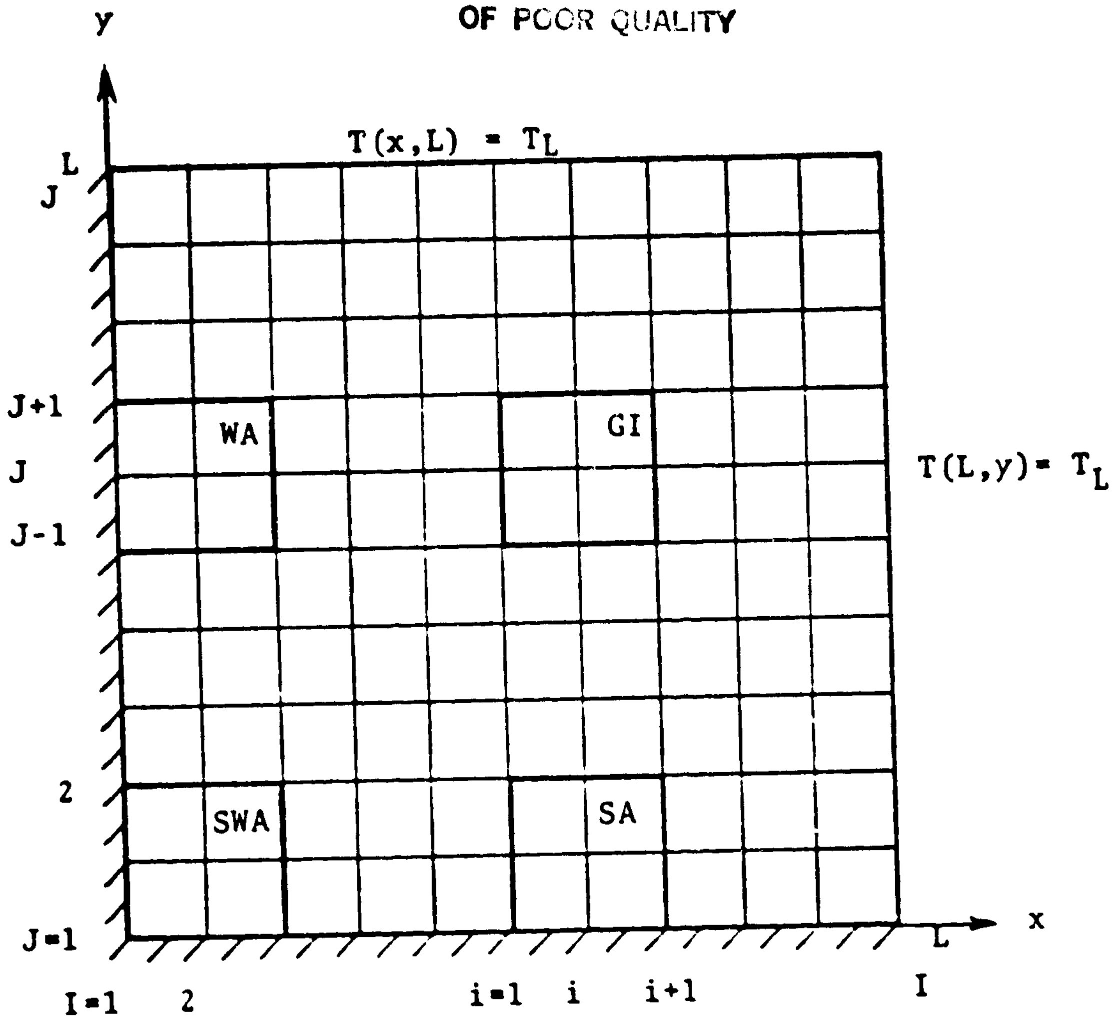

Figure (V-1). Nodal-Point Arrangement for Two Dimensional Steady Heat Conduction in a Square Region with Uniform Heat Generation.

V.1 Finite Analytic Solution of the Problem

The first step in the FA solution to the problem is to subdivide the region into a finite number of subregions by setting up a system of nodes as shown in Figure (V-1). As shown in Figure (V-1),the subdivided problem involves three distinct types of subregions named General Interior (GI), SouthWest-Adiabatic (SWA), and South-Adiabatic (SA) or West-Adiabatic (WA). The boundary conditions for these three distinct types of subregions as shown in Figures (IV-2), (IV-3), and (IV-4) are following.

Type (GI),

$$y = 0 \qquad \psi = f_S(x),$$

$$y = 2k \qquad \psi = f_N(x),$$

$$x = 0 \qquad \psi = f_W(y),$$

$$x = 2h \qquad \psi = f_E(y). \tag{V-5}$$

Type (SWA),

$$y = 0 \qquad \frac{\partial\psi}{\partial y} = 0,$$

$$y = 2k \qquad \psi = f_N(x),$$

$$x = 0 \qquad \frac{\partial\psi}{\partial x} = 0,$$

$$x = 2h \qquad \psi = f_E(y). \tag{V-6}$$

Type (SA)

$$y = 0 \qquad \frac{\partial\psi}{\partial y} = 0$$

$$y = 2k \qquad \psi = f_N(x)$$

$$x = 0 \qquad \psi = f_W(y)$$

$$x = 2h \qquad \psi = f_E(y) \tag{V-7}$$

where the local boundary condition f's have already been described in Chapter IV.

V.1.1 The Formulation of the FA Method for ψ_p

The basic principle of the FA method and the 9-point FA formulas for the Poisson equation in three types of sub-regions GI, SA, and SWA Tables (IV-1), (IV-3) and (IV-4) were already presented and derived in the previous chapter. However, if the temperature along the insulated boundaries desired, the local analytic solution derived in the previous Chapter can be easily evaluated at these nodes. This is given in the following section.

V.1.2 The 8-point FA Formulas for the Nodal Values along the Adiabatic Boundaries

Consider the Figures (V-1),(IV-2), (IV-3) and (IV-4). There are three distinct boundary nodes as SC in subregion (SA), SC and SW in subregion (SWA). The 8-point FA formulas for the nodes along the insulated boundaries can be obtained from the analytic solution in each subregion evaluated at the boundary nodes.

The 8-point FA formula for the boundary node SC in subregion (SA) is

$$\psi_{SC} = C_{EC}\,\psi_{EC} + C_{WC}\,\psi_{WC} + C_{NC}\,\psi_{NC} + C_{NE}\,\psi_{NE} + C_{SW}\,\psi_{SW}$$

$$+ \; C_{SE}\,\psi_{SE} + C_{NW}\,\psi_{NW} + C_{GP} \qquad \text{(V-8)}$$

where

$$C_{EC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a)\right] \frac{\sinh(a\gamma/2)}{\sinh(a\gamma)},$$

$$C_{WC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a)\right]\left[\cosh\left(\frac{a\gamma}{2}\right) - \frac{\sinh(a\gamma/2)}{\tanh(a\gamma)}\right],$$

$$C_{NC} = \sum_{n=1}^{\infty} \frac{4}{b^3}\,\frac{\sin(b)}{\cosh(b\gamma)},$$

$$C_{NE} = \sum_{n=1}^{\infty} \left[\frac{2}{a^2} + \left(\frac{2}{a} - \frac{8}{a^3}\right)\sin(a)\right] \frac{\sinh(a\gamma/2)}{\sinh(a\gamma)} +$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \frac{\sin(b)}{\cosh(b/\gamma)},$$

$$C_{SW} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3}\sin(a)\right]\left[\cosh\left(\frac{a\gamma}{2}\right) - \frac{\sinh(a\gamma/2)}{\tan(a\gamma)}\right],$$

$$C_{SE} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3}\sin(a)\right] \frac{\sinh(a\gamma/2)}{\sinh(a\gamma)},$$

$$C_{NW} = \sum_{n=1}^{\infty} \left[\frac{2}{a^2} + \left(\frac{2}{a} - \frac{8}{a^3}\right)\sin(a)\right]\left[\cosh\left(\frac{a\gamma}{2}\right) - \frac{\sinh(a\gamma/2)}{\tanh(a\gamma)}\right]$$

$$+ \sum_{n=1}^{\infty} \left(\frac{1}{b} - \frac{2}{b^3}\right) \frac{\sin(b)}{\cosh(b/\gamma)},$$

$$C_{GP} = \sum_{n=1}^{\infty} \frac{2b^2}{b^3}\left[1 - \frac{1}{\cosh(2b/\gamma)}\right]\sin(b).$$

Again, for $\gamma = 1$ the universal 8-point FA formula for sub-region (SA) can be obtained. The numerical values of these coefficients are listed in Table (V-1).

The 8-point formula for boundary node SC in sub-region (SWA) is equation (V-8) where

$$C_{EC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a)\right] \frac{\cosh(a\gamma/2)}{\cosh(a\gamma)},$$

$$C_{WC} = 0,$$

$$C_{NC} = \sum_{n=1}^{\infty} \left[-\frac{8}{a^2} + \frac{16}{a^3}\sin(a)\right] \frac{\cosh(a\gamma/2)}{\cosh(a\gamma)},$$

$$C_{NE} = \sum_{n=1}^{\infty} \left[\frac{2}{a^2} + \left(\frac{2}{a} - \frac{8}{a^3}\right)\sin(a)\right]\left[\frac{\cosh(a\gamma/2)}{\cosh(a\gamma)} + \frac{\cos(a/2)}{\cosh(a/\gamma)}\right],$$

$$C_{SW} = 0$$

$$C_{SE} = \sum_{n=1}^{\infty} [\frac{6}{a^2} - \frac{8}{a^3} \sin(a)] \frac{\cosh(a\gamma/2)}{\cosh(a\gamma)},$$

$$C_{NW} = \sum_{n=1}^{\infty} [\frac{6}{a^2} - \frac{8}{a^3} \sin(a)] \frac{\cos(a/2)}{\cosh(a/\gamma)},$$

$$C_{GP} = \sum_{n=1}^{\infty} \frac{8k^2}{a^3} \sin(a) [1 - \frac{1}{\cosh(a/\gamma)}] \cos(a/2).$$

The numerical values of these coefficients for $\gamma = 1$ are presented in Table (V-2).

The 8-point FA formula for node SW in subregion (SWA) is

$$\psi_{SC} = C_{EC}\,\psi_{EC} + C_{WC}\,\psi_{WC} + C_{NC}\,\psi_{NC} + C_{SC}\,\psi_{SC} + C_{NE}\,\psi_{NE}$$

$$+ C_{SE}\,\psi_{SE} + C_{NW}\,\psi_{NW} + C_{GP} \qquad (V\text{-}9)$$

where

$$C_{EC} = \sum_{n=1}^{\infty} [-\frac{8}{a^2} + \frac{16}{a^3} \sin(a)]/\cosh(a\gamma),$$

$$C_{WC} = 0$$

$$C_{NC} = \sum_{n=1}^{\infty} [-\frac{8}{a^2} + \frac{16}{a^3} \sin(a)]/\cosh(a/\gamma),$$

$$C_{SC} = 0$$

ψ_{SC} =

NW -0.001627	NC 0.089025	NE -0.001627
WC 0.283945		EC 0.283945
SW 0.173170		SE 0.173170

$x\psi_n$ +

$h^2(0.455731)$

Table (V-1). FA Numerical Values for Coefficients in a Subregion with One Insulated Boundary for the Point on the Insulated Boundary.

ψ_{SC} =

NW	NC	NE
0.099149	0.256229	-0.000786
WC 0.0		EC 0.416351
SW 0.0		SE 0.229057

$x\psi_n^+$

$k^2(0.917533)$

Table (V-2). FA Numerical Values for Coefficients in a Subregion with Two Insulated Boundaries for the Point on the Insulated Boundary.

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$$C_{NE} = \sum_{n=1}^{\infty} \left[\frac{2}{a^2} + \left(\frac{2}{a} - \frac{8}{a^3}\right) \sin(a)\right]\left[\frac{1}{\cosh(a\gamma)} + \frac{1}{\cosh(a/\gamma)}\right],$$

$$C_{SE} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3} \sin(a)\right]/\cosh(a\gamma),$$

$$C_{NW} = \sum_{n=1}^{\infty} \left[\frac{6}{a^2} - \frac{8}{a^3} \sin(a)\right]/\cosh(a/\gamma),$$

$$C_{GP} = \sum_{n=1}^{\infty} \frac{8h}{a^3} \sin(a) \left[1 - \frac{1}{\cosh(a/\gamma)}\right].$$

The numerical values of these coefficients for $\gamma = 1$, are listed in Table (V-3). It should be mentioned that for the nodes on the west boundary side in subregion WA, the FA formulas are obtained by rotating the coordinates of the south side.

V.2 Numerical Solution of the Problem

Let us now apply the finite analytic formula to the example with equation (V-3) and (V-4). Consider the FA solution with uniform grid size h = k as shown in Figure (V-1). The grid spacing h, for n number of grids is equal to $\frac{L}{n}$, with the origin (0,0) (which corresponds to i=1, J=1) at the south west corner and a grid size h as shown in Figure (V-1), the numbering on the y-coordinate will be J = 1,2,3...J, and on the x-coordinate i=1,2,3...I, respectively. The purpose of the FA method is to find a numerical solution ψ at points J = 1,2,...J and i=1,2,...I from equations (V-3) and (V-4).

ψ_{SW} =

NW 0.157798	NC 0.343775	NE 0.006854
WC 0.0		EC 0.343775
	SC 0.0	SE 0.152798

$x\psi_n + h^2(1.178614)$

Table (V-3). FA Numerical Values for Coefficients in a Subregion with Two Insulated Boundaries for the Point on the Corner (SW).

The finite analytic scheme results with a set of simultaneous, linear, algebraic equations to solve. Thus, there are nxn unknown nodal temperature values to be solved. The nxn FA algebraic equations can be constructed from the 9-point and 8-point FA formulas (Tables IV-1, IV-3, IV-4, and V-1 to V-3). The FA equations can be cast in the following forms.

For any general internal subregion, (i>2, J>2)

$$\psi_{i,J}^{m+1} = .205315\ (\psi_{i,J+1}^{m} + \psi_{i+1,J}^{m} + \psi_{i,J-1}^{m} + \psi_{i-1,J}^{m})$$

$$+ .044685\ (\psi_{i+1,J+1}^{m} + \psi_{i+1,J-1}^{m} + \psi_{i-1,J-1}^{m}$$

$$+ \psi_{i-1,J+1}^{m}) + h^2\ (0.29493) \qquad \text{(V-10)}$$

for the nodes along the y = 0, (i=1,J>2)

$$\psi_{1,J}^{m+1} = .283945\ (\psi_{2,J+1}^{m} + \psi_{2,J-1}^{m}) + .173170\ (\psi_{1,J+1}^{m}$$

$$+ \psi_{1,J-1}) - .001627\ (\psi_{3,J+1}^{m} + \psi_{3,J-1}^{m}) +$$

$$.089025\ \psi_{3,J}^{m} + h^2 (0.455731) \qquad \text{(V-11)}$$

In the similar fashion the equation along x = 0, (i>2, J=1)

$$\psi^{m+1}_{i,1} = .283945\ (\psi^m_{i-1,2} + \psi^m_{i+1,2}) + .173170\ (\psi^m_{i+1,1} + \psi^m_{i-1,1}) - .001627\ (\psi^m_{i-1,3} + \psi^m_{i+1,3}) + .089025\ \psi^m_{i,3} + h^2(0.455731) \qquad \text{(V-12)}$$

At point $(x,y) = (0,0)$ the equation is given

$$\psi^{m+1}_{i,1} = .152798\ (\psi^m_{1,3} + \psi^m_{3,1}) + .343775\ (\psi^m_{2,3} + \psi^m_{3,2}) + .006854\ \psi^m_{3,3} + h^2(1.178614) \qquad \text{(V-13)}$$

At point $(x,y) = (2,2)$ the equation is

$$\psi^{m+1}_{2,2} = .112834\ (\psi^m_{1,3} + \psi^m_{3,1}) + .366502\ (\psi^m_{2,3} + \psi^m_{3,2}) + .041327\ \psi^m_{3,3} + h^2(0.724753) \qquad \text{(V-14)}$$

For the points $(i=2,J>2)$ and $(i>2,J=2)$, the equations are respectively

$$\psi^{m+1}_{2,J} = .271649\ (\psi^m_{2,J+1} + \psi^m_{2,J-1}) + 0.074144\ (\psi^m_{1,J+1} + \psi^m_{i,J-1}) + .042678\ (\psi^m_{3,J+1} + \psi^m_{3,J-1}) + .223055\ \psi^m_{3,J} + h^2(0.388716) \qquad \text{(V-15)}$$

$$\psi^{m+1}_{i,2} = .271649\ (\psi^m_{i+1,2} + \psi^m_{i-1,2}) + .074144\ (\psi^m_{i+1,1} + \psi^m_{i-1,1}) + .042678\ (\psi^m_{i+1,3} + \psi^m_{i-1,3}) + + .223055\ \psi^m_{i,3} + h^2(0.388716) \qquad \text{(V-16)}$$

Finally, for the points (i=1,J=2) and (i=2,J=1) the formulas are

$$\psi^m_{1,2} = .229057\ \psi^m_{1,3} + .416351\ \psi^m_{2,3} - .000786\ \psi^m_{3,3} + .256229\ \psi^m_{3,2} + .099149\ \psi^m_{3,1} + h^2(0.917533) \qquad \text{(V-17)}$$

$$\psi^m_{2,1} = .229057\ \psi^m_{3,1} + .416351\ \psi^m_{3,2} - .000786\ \psi^m_{3,3} + .256229\ \psi^m_{2,3} + .099149\ \psi^m_{1,3} + h^2(0.917533) \qquad \text{(V-18)}$$

In this particular example the temperature at each node on the north and east boundaries is known, making it unnecessary to write special equations at the two boundaries. Equations (V-10) to (V-18) represent the system of algebraic equations that must be solved for the unknown nodal temperatures.

An iterative numerical procedure can be used to solve the system of linear equations within some tolerance ε.

V.2.1 The Iterative Method of Solution

The application of the finite analytic method to the problem has now resulted in a set of simultaneous, linear, algebraic equations. As an illustration, let us consider the grid size h to be equal to 1/4. This nodal point arrangement is shown in Fig. (V-2). Observe that the nodal points (1,5),....,(5,1) along the north and east boundaries are all at zero temperature as given by the boundary conditions of the problem. Thus, there are only sixteen unknown temperatures. The Gauss-Siedel method is applied to solve this system of equations. The computational procedure begins with initial guesses $\psi_{i,j}^{(0)}$ for all the unknowns. An improved value for each of the unknowns $\psi_{i,j}^{(1)}$ is then computed from Eqs. (V-10) to (V-18). This iterative process can be carried out until it converges. That is the difference between two iteratations within a required accuracy or $|\psi_{i,j}^{m+1} - \psi_{i,j}^{m}| < \varepsilon$, where the value of ε sets the error criterion. For example in this problem, $\psi_{1,1}$ requires thirty-nine iterations to converge into four significant figures.

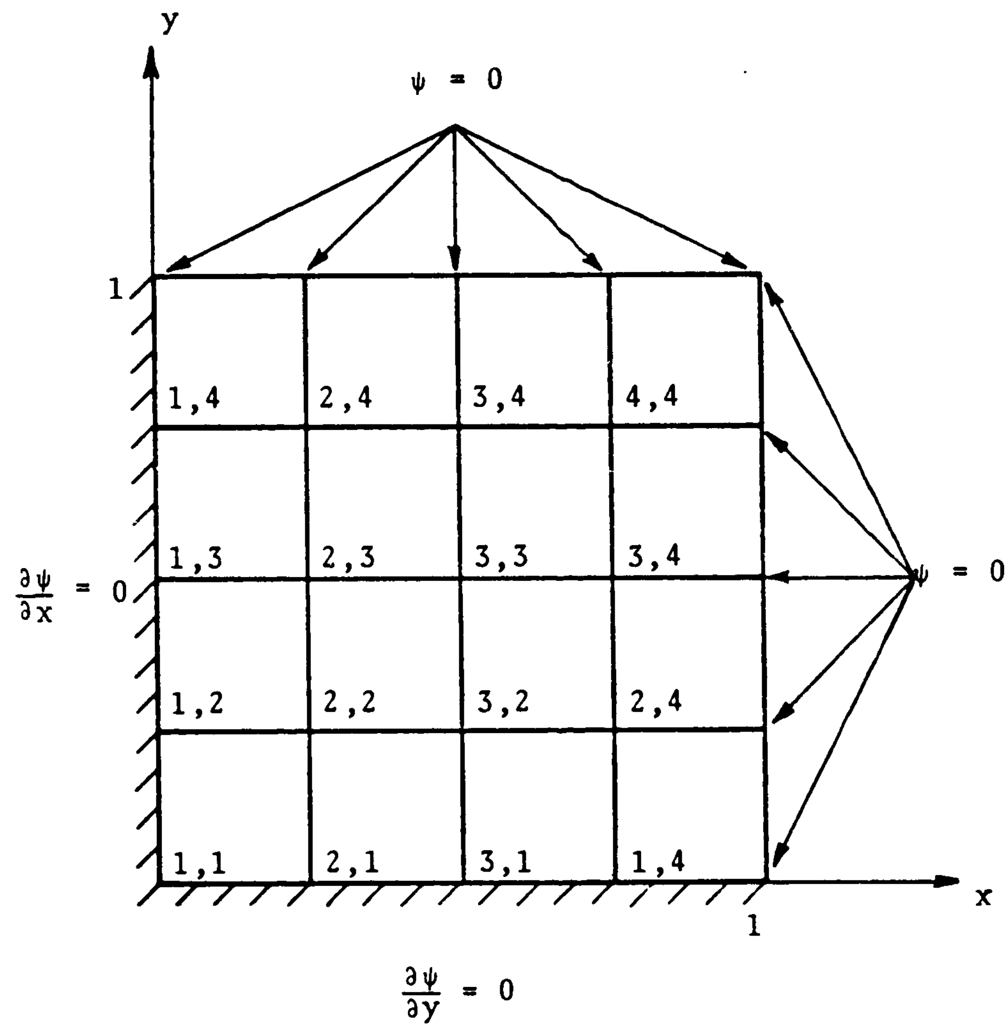


Figure (V-2). The Region with the Corresponding Boundary Conditions where the Equation is Solved for Comparison of Different Numerical Methods.

A method to accelerate the convergence of the iteration process is the method of successive overrelaxation (SOR). In this technique the update value $\psi^{m+1}_{i,j}$ at a given node is replaced by the following equation.

$$\psi^{m+1}_{i,j} = \psi^{m}_{i,j} + w(\bar{\psi}^{m+1}_{i,j} - \psi^{m}_{i,j}) \tag{V-19}$$

where $\bar{\psi}^{m+1}_{i,j}$ is the value just calculated. The relaxation factor w may be thought of as a weighting factor. For $w = 1$ the new value of ψ would be the same as calculated in the Gauss-Siedel procedures. The method is underrelaxed if $o < w < 1$ and is overrelaxed if $w > 1$. In this particular example, an overrelaxation value of $w = 1.4$ is used and the solution at $\psi_{1,1}$ converges into four significant figures after fourteen iterations instead of the thirty-nine iterations required for the Gauss-Siedel method.

V.2.2 Numerical Results

We will now show the numerical solutions of the same problem using finer grid size with $h = 1/4$ as shown in Figure (V-2).

In order to discuss the finite analytic (FA) solution, the problem given in this chapter is also solved by finite difference (FD) and finite element (FE) methods which are given by Mayer [17]. The numerical solution of the problem and errors using FA, FD, and FE numerical methods are listed in Tables (V-4) to (V-6). In all three tables the first column is the location of the node as shown in Figure (V-2). The second column is the exact solution of the problem, the third column shows the numerical solution, and the fourth column states the error of the method. The finite difference solution given by Mayer [17] is based on 5-point central difference.

The finite element solution given in Table (V-6) is based on variational formulation of the differential equation. In this two dimensional conduction problem a three nodal-right triangular finite element is used. It is assumed that the temperature varies linearly between the three corner temperatures. To compare the finite analytic method with the finite element method it will be instructive to look at the finite element equations for a nodal spacing of 1/4.

From the tables it is observed that the finite analytic solution gives more accurate solution than the other methods. For example, at the node (2,3) the error of FA solution is accurate to 10^{-5} while the finite difference and the finite element solutions have errors of 0.0027 and 0.0006 respectively. At node (1,1) the error for FA solution is -0.0001 while the error for the finite difference and finite element solutions are 0.0036 and -0.0066 respectively. It should be remarked that the finite analytic solution does not have the truncation error as in the finite difference approximation. The only approximation made in the finite analytic solution is that the boundary functions $f_E(y)$, $f_S(x)$, $f_N(x)$, and $f_W(y)$ are approximated with a second degree polynomial. An improved FA solution may be obtained if each subregion is made to have five nodes on the boundary shown on dashed lines in Figure (III-2). In this case the boundary functions f_E, f_S, f_N, and f_W are approximated by a polynomial of fourth degree, for example

$$f_E(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 \qquad \text{(V-20)}$$

It should be mentioned that the FA solution is less sensitive to the grid size than the other methods.

Node	Exact	Finite Analytic	Error Exact-Approx
1,1	0.2947	0.2948	-0.0001
1,2	0.2789	0.2790	-0.0001
1,3	0.2293	0.2294	-0.0001
1,4	0.1397	0.1398	-0.0001
2,1	0.2789	0.2790	-0.0001
2,2	0.2642	0.2642	0.0
2,3	0.2178	0.2178	0.0
2,4	0.1333	0.1333	0.0
3,1	0.2293	0.2294	-0.0001
3,2	0.2178	0.2178	0.0
3,3	0.1811	0.1811	0.0
3,4	0.1127	0.1126	0.0001
4,1	0.1397	0.1398	-0.0001
4,2	0.1333	0.1333	0.0
4,3	0.1127	0.1126	0.0001
4,4	0.0728	0.0727	0.0001

Table (V-4). Finite Analytic Solution for Square with Heat Generation.

Node	Exact	Finite Difference (5-point)	Error Exact-Approx
1,1	0.2947	0.2911	0.0036
1,2	0.2789	0.2755	0.0034
1,3	0.2293	0.2266	0.0027
1,4	0.1397	0.1381	0.0016
2,1	0.2789	0.2755	0.0034
2,2	0.2642	0.2609	0.0033
2,3	0.2178	0.2151	0.0027
2,4	0.1333	0.1317	0.0016
3,1	0.2293	0.2266	0.0027
3,2	0.2178	0.2151	0.0027
3,3	0.1811	0.1787	0.0024
3,4	0.1127	0.1110	0.0017
4,1	0.1397	0.1381	0.0016
4,2	0.1333	0.1317	0.0016
4,3	0.1127	0.1110	0.0017
4,4	0.0728	0.0711	0.0017

Table (V-5). Finite Difference Solution for Square with Heat Generation.

Node	Exact	Finite Element	Error Exact-Approx
1,1	0.2947	0.3013	-0.0066
1,2	0.2789	0.2805	-0.0016
1,3	0.2293	0.2292	0.0001
1,4	0.1397	0.1392	0.0005
2,1	0.2789	0.2805	-0.0016
2,2	0.2642	0.2645	-0.0003
2,3	0.2178	0.2172	0.0006
2,4	0.1333	0.1327	0.0006
3,1	0.2293	0.2292	0.0001
3,2	0.2178	0.2172	0.0006
3,3	0.1811	0.1801	0.0010
3,4	0.1127	0.1117	0.0010
4,1	0.1397	0.1392	0.0005
4,2	0.1333	0.1327	0.0006
4,3	0.1127	0.1117	0.0010
4,4	0.0728	0.0715	0.0013

Table (V-6). Finite Element Solution for Square with Heat Generation.

CHAPTER VI

APPLICATION OF FINITE ANALYTIC METHOD TO THE LAPLACE EQUATION WITH COMPLEX GEOMETRY

In this chapter a new procedure of implimenting the FA method is given. This procedure is for the problem which has a relatively simple partial differential equation and could be solved analytically. For example, finding an analytic solution for a steady heat conduction with constant conductivity, which is governed by the Laplace equation in a problem with irregular geometry as shown in Figure (VI-1), is almost impossible. Following, it will be shown that the finite analytic method may be implemented quite differently from the procedures used in the previous chapters to solve the problem. Consider Figure (VI-1) which shows the cross section of a groove bounded by two slabs. Let the dimensionless temperature on top including the groove walls be one and the bottom surfaces be normalized to zero. Let the temperature at the side walls of the slabs vary linearly from zero at the bottom surface to one at the top. This problem will be solved by the FA method for different sizes of groove and slabs dimensions. In some engineering designs it is important to know the effect of the groove sizes on the temperature distortion and to calculate the heat flux at the bottom surface.

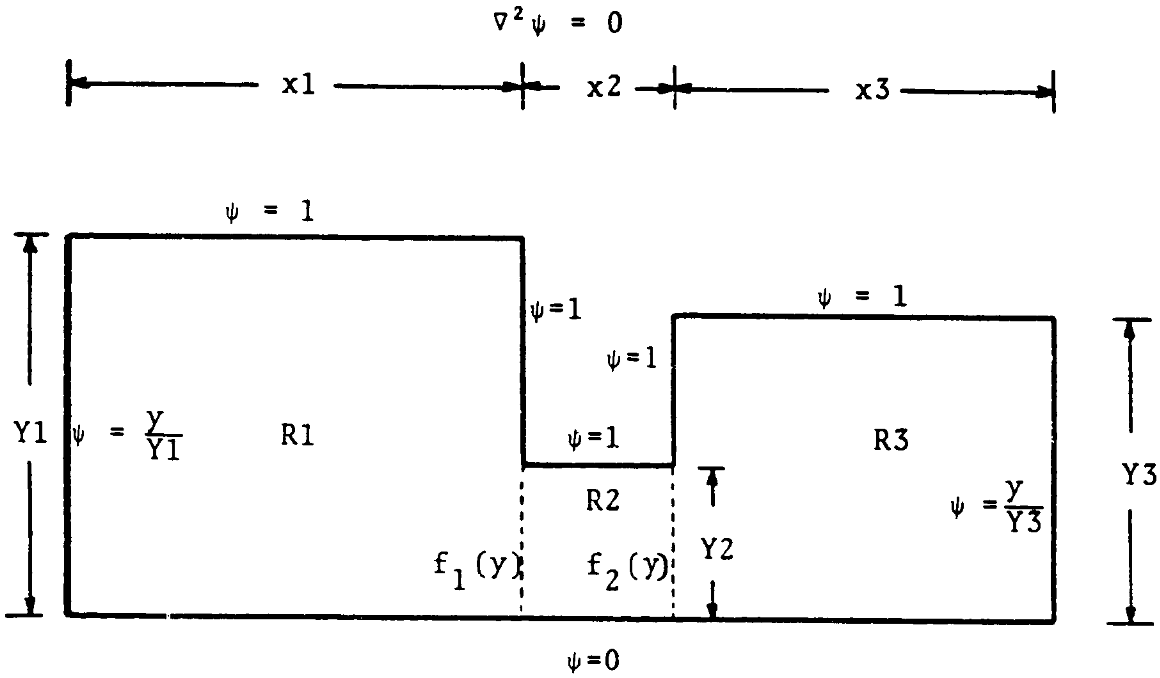


Figure (VI-1). The Cross Section of a Groove in the Slab.

VI.1 The Method of Solution

The finite analytic numerical solution to this problem can be solved by two different procedures. The first is to subdivide the problem into many subregions (as was done in previous chapters) and solve every nodal value numerically. The second is to subdivide the problem only into three rectangular regions, R1, R2, and R3 as shown in Figure (VI-2) where only 2N nodal points are assigned to the common boundaries of the regions R1 and R2 and regions R2 and R3. In this case, the analytic solution for each region can be obtained by separating variables once the boundary conditions on the common boundaries of the region are specified. The common boundary conditions may be approximated by a function of y (or a set piecewise continuous function) and the unknown nodal values specified at N nodal points. That is, the temperature functions along the boundaries between R1 and R2 and R2 and R3, $f_1(y)$ and $f_2(y)$ are approximated respectively by the functions in terms of the nodal temperatures ψ_1, ψ_2, $\ldots\psi_N$ and y and

ψ_1', ψ_2', ..., ψ_N' and y such as

$$
\begin{aligned}
f_1(y) &= f_1(\psi_1, \psi_2, \ldots, \psi_N, h, y) \\
f_2(y) &= f_2(\psi_1', \psi_2', \ldots, \psi_N', h, y)
\end{aligned}
\qquad \text{(VI-1)}
$$

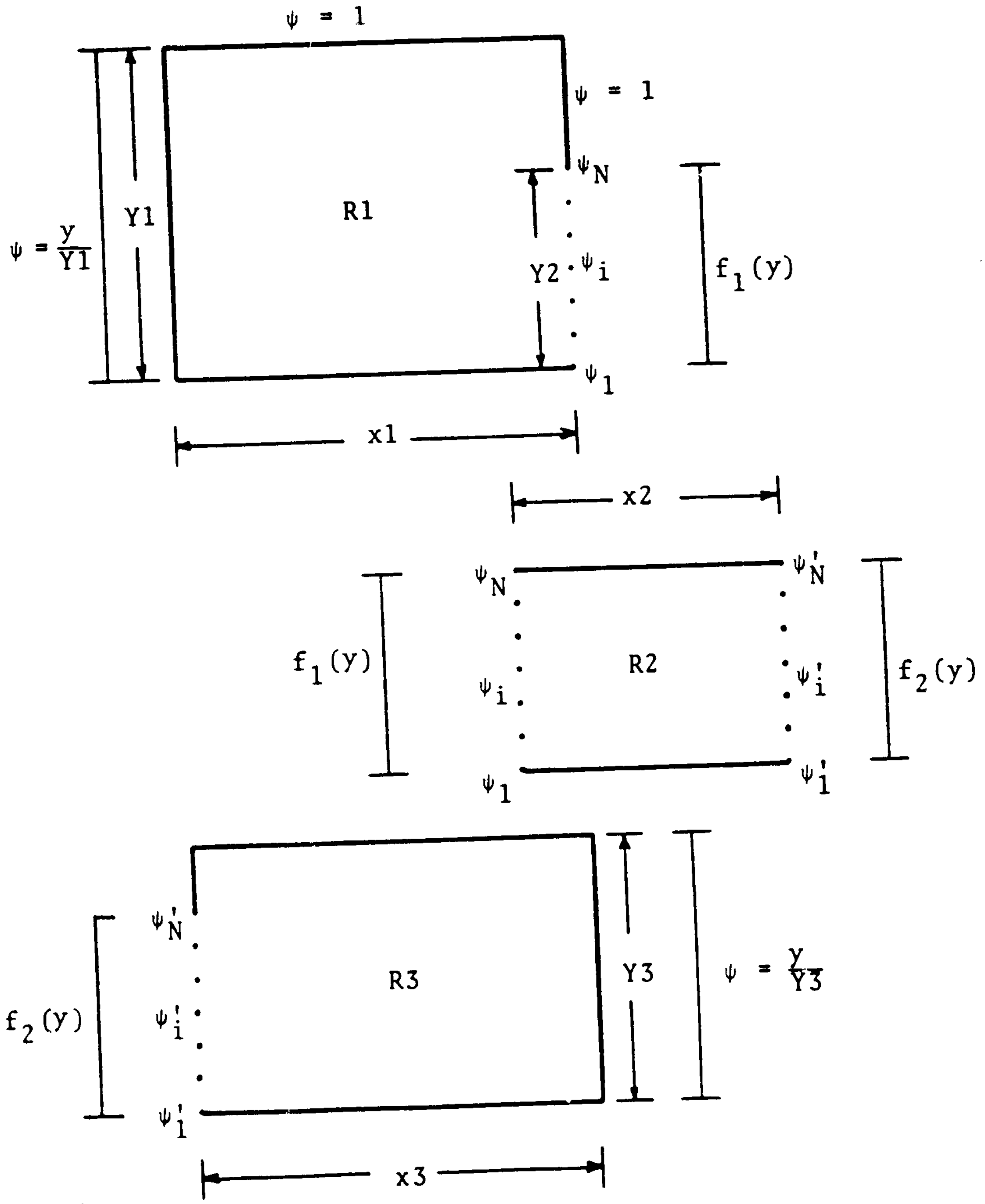


Figure (VI-2). Three Regions R1, R2, and R3 with their Corresponding Boundary Conditions.

where h is the grid size and ψ_1, $\psi_2,\ldots,\psi_N$ and ψ_1', $\psi_2',\ldots,\psi_N'$ are boundary nodal points.

The temperature distribution in each of the rectangular regions ψ_{R1}, ψ_{R2}, and ψ_{R3} can now be obtained by solving the Laplace equation with the corresponding boundary conditions. Thus,

$$\psi_{R1} = f_{R1}(\psi_1,\ \psi_2,\ldots,\psi_N,h,x,y) \tag{VI-2a}$$

$$\psi_{R2} = f_{R2}(\psi_1,\ \psi_2,\ldots,\psi_N,\psi_1',\ \psi_2',\ldots,\psi_N',h,x,y) \tag{VI-2b}$$

$$\psi_{R3} = f_{R3}(\psi_1',\ \psi_2',\ldots,\psi_N',h,x,y) \tag{VI-2c}$$

Each of the above solutions is the analytic solution respectively to the regions R1, R2, and R3. However, only where the unknown temperatures ψ_1, $\psi_2,\ldots,\psi_N$ and ψ_1', $\psi_2',\ldots,\psi_N'$ are determined, the equations (VI-2) provide the solution for the entire region of the problem.

There are several ways of finding the unknowns ψ_1, $\psi_2,\ldots,\psi_N$ and ψ_1', $\psi_2',\ldots,\psi_N'$. One of them is to generate 2N independent algebraic equations from the matching condition that requires either the temperature or the temperature gradient (heat flux) must be continous at each of the common boundary points.

In this present work the FA method is applied to finding the unknowns ψ_1, $\psi_2, \ldots, \psi_N$ and ψ'_1, $\psi'_2, \ldots, \psi'_N$. The solution's procedures are described as follows: As the first step to attaining the solution, we select 2N nodal points on the common boundaries (N nodal points on each common line). Then, many square subregions (2hx2h) can be constructed along each common boundary as shown in Figure (VI-3). The unknown temperatures ψ_2, $\psi_3, \ldots, \psi_{N-1}$ and ψ'_2, $\psi'_3, \ldots, \psi'_{N-1}$ on each common boundary are the interior nodal points of these subregions. A typical subregion around an interior node, located at the point (xl, y_n) along the common boundary between R1 and R2, is shown in Figure (VI-4). TW's and TE's are the temperatures on the west and east sides of the subregions respectively. In the subregions along the common boundary between R2 and R3, ψ'_2, $\psi'_3, \ldots, \psi'_{N-1}$ are the interior nodal points as well as the TW′s and TE′s are temperatures on the subregions' east and west boundaries. For each element, a 9-point FA solution, such as given in Table (IV-1), can be derived to relate the centeral nodal value ψ_n to the surrounding nodal values TE(n+1), TE(n), TE(n-1), TW(n+1), TW(n), TW(n-1), ψ_{n+1} and ψ_{n-1}. For instance, in the Laplace equation one has:

$$\psi_n = 0.044685\ [TE(n+1)+TE(n-1)+TW(n+1)+TW(n-1)] + 0.2051315\ [TE(n)+TW(n) + \psi_{n-1} + \psi_{n+1}] \qquad \text{(VI-3)}$$

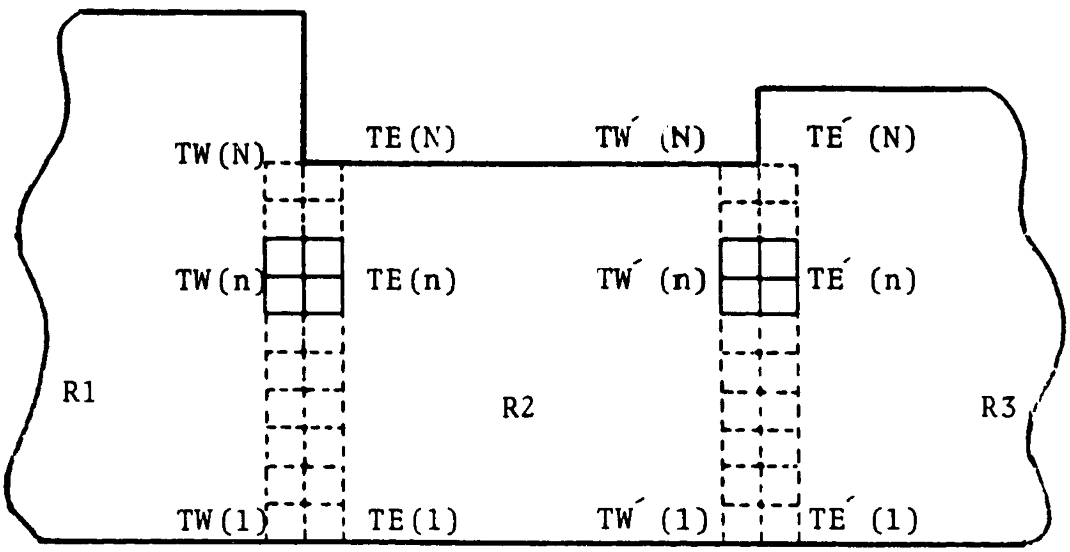


Figure (VI-3). Nodal-Point Arrangement on the Common Boundaries.

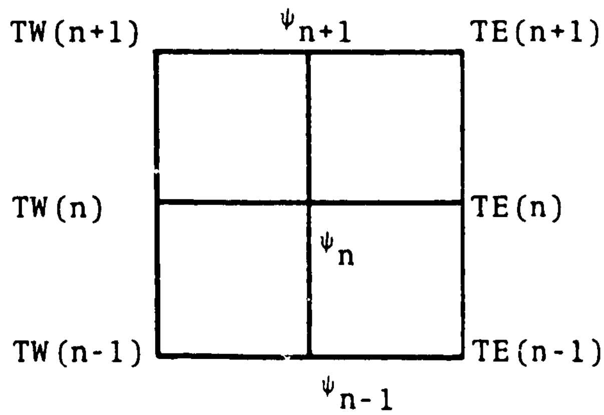


Figure (VI-4). A Typical Subregion Around an Interior Node ψ_n on the Common Boundary.

This finite analytic solution can be repeated for all nodal points on the common boundaries. It should be noted that the nodal values TE's TW's can be found from the analytic solutions given in equations (VI-2) in terms of the unknown boundary nodal values ψ_n or ψ'_n. An iterative procedure may be set up to solve the unknown nodal values ψ_n and ψ'_n. That is to guess $\psi_n^{(0)}$ and $\psi_n'^{(0)}$ (for all unknowns) as the initial trial values for $\psi_2, \psi_3, \ldots, \psi_{N-1}$ and $\psi'_2, \psi'_3, \ldots, \psi'_{N-1}$, then equations (VI-2) provide the solution for any point like, TE, TW, TE′ and TW′ in the whole problem, which means the initial guesses for ψ_n and ψ'_n will give us the initial values, $TE(n)^{(0)}$, $TW(n)^{(0)}$, $TE'(n)^{(0)}$ and $TW'(n)^{(0)}$. Using the 9-point FA formula, as was mentioned above, equation (VI-3) will give us the new values for interior nodal points ψ_n and ψ'_n, or:

$$\psi_n^{m+1} = 0.44685\ [\overset{m}{TE}(n+1)+\overset{m}{TE}(n-1)+\overset{m}{TW'}(n+1)+\overset{m}{TW'}(n+1)] + 0.2051315\ [\overset{m}{TE}(n)+\overset{m}{TW}(n) + \psi_{n-1}^m + \psi_{n+1}^m] \quad \text{(VI-4)}$$

$$\psi_n'^{m+1} = 0.44685\ [\overset{m}{TE'}(n+1)+\overset{m}{TE'}(n-1)+\overset{m}{TW'}(n+1)+\overset{m}{TW'}(n-1)] + 0.2051315\ [\overset{m}{TE'}(n)+\overset{m}{TW'}(n) + \psi_{n-1}'^m + \psi_{n+1}'^m] \quad \text{(VI-5)}$$

This iterative process can be repeatedly carried out until it converges. That is $|\psi_n^{m+1} - \psi_n^m| < \varepsilon$ where ε is the convergence criterion.

VI.1.1 The FA Solution for Subregions R1, R2, and R3

In order to have an analytic solution for each subregion R1, R2, and R3 let us first approximate the boundary condition $f_1(y)$ and $f_2(y)$ respectively by a piecewise polynomial (i.e., segmental polynomial) so that the problem may employ any arbitrary number of nodal points. For simplicity and flexibility, a set of piecewise second degree polynomials in finite subintervals is chosen to represent the $f_1(y)$ and $f_2(y)$. Therefore, the function $P_i(y')$ is a polynomial of the second degree on the subintervals (ψ_1, ψ_2, ψ_3), (ψ_3, ψ_4, ψ_5),.....,$(\psi_{N-2}, \psi_{N-1}, \psi_N)$ shown in Figure (VI-5), or:

$$P_i(y') = C_{0i} + C_{1i}y' + C_{2i}y'^2 \qquad \text{(VI-6)}$$

where

$$C_{0i} = \psi_{i-2}$$

$$C_{1i} = -\frac{3}{2h}\psi_{i-2} + \frac{2}{h}\psi_{i-1} - \frac{1}{2h}\psi_i$$

$$C_{2i} = \frac{1}{2h^2}\psi_{i-2} - \frac{1}{h^2}\psi_{i-1} + \frac{1}{2h^2}\psi_i$$

The boundary function $f_1(y)$ on each interval $(\psi_{i-1,2}, \psi_{i-1}, \psi_i)$ may be written as

$$f_1(y) = P_i(y') \quad (i-2)h \le y \le ih \tag{VI-7}$$

Similarly, the boundary $f_2(y)$ may be written as

$$f_2(y) = P_i'(y'), \quad (i-2)h \le y \le ih \tag{VI-8}$$

where $P_i'(y')$ is a quadratic polynomial on each interval $(\psi_{i-2}', \psi_{i-1}', \psi_i')$. After specifying $f_1(y)$ and $f_2(y)$, the analytic solutions ψ_{R1}, ψ_{R2} and ψ_{R3} for subregions R1, R2, and R3 may be obtained as follows:

Region R1: $\nabla^2 \psi_{R1} = 0$ (VI-9)

$$x = 0, \qquad \psi = \frac{y}{Y1}$$

$$x = x1, \qquad \psi = \begin{cases} f_1(y) & Y2 > y \ge 0 \\ 1 & y \ge Y2 \end{cases}$$

$$y = 0, \qquad \psi = 0$$

$$y = Y1, \qquad \psi = 1$$

Region R2: $\nabla^2 \psi_{R2} = 0$ (VI-10)

$$x = 0, \qquad \psi = f_1(y)$$

$$x = x2, \qquad \psi = f_2(y)$$

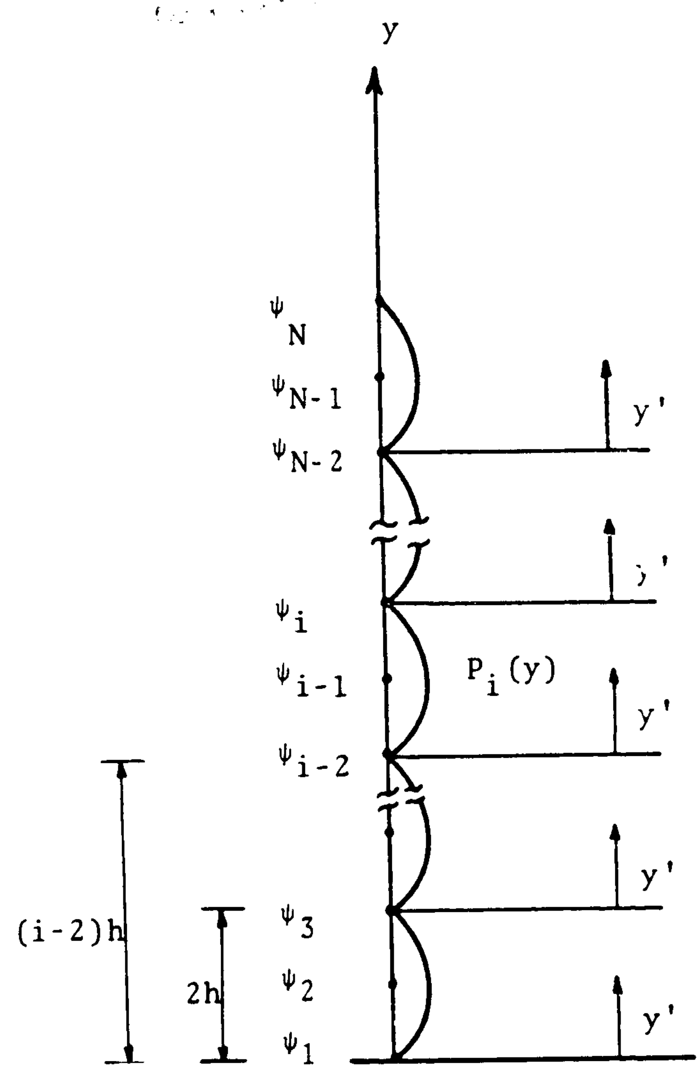


Figure (VI-5). The Functional Approximation for the Common Boundaries.

$$y = 0, \qquad \psi = 0$$

$$y = Y2, \qquad \psi = 1$$

Region R3: $$\nabla^2 \psi_{R3} = 0 \qquad \text{(VI-11)}$$

$$x = 0, \qquad \psi = \begin{cases} f_2(y) & Y2 > y \geq 0 \\ 1 & y \geq Y2 \end{cases}$$

$$x = x3, \qquad \psi = \frac{y}{Y3}$$

$$y = 0, \qquad \psi = 0$$

$$y = Y3, \qquad \psi = 1$$

The above problems are solved by the separation of variables. The solutions ψ_{R1}, ψ_{R2} and ψ_{R3} involve only 2N-4 unknown nodal values $\psi_2, \psi_3, \ldots, \psi_{N-1}$ and $\psi'_2, \psi'_3, \ldots, \psi'_{N-1}$ since the nodal values ψ_1, ψ_N, ψ'_1 and ψ'_N are known or $\psi_1 = \psi'_1 = 0$ and $\psi_N = \psi'_N = 1$. It should be noticed that if the 2N-4 unknowns along the common boundaries are known the solution for the temperature distribution of the problem is found.

VI.1.2 The FA Solution for Subregion R1

The analytic solution ψ for region R1 with governing and boundary conditions (equation (VI-9)) can be obtained by the separation of variables. Since the Laplace equation is linear, the solution to the subregion can be superposed by three solutions with simpler boundary conditions:

$$\psi_{R1} = \psi_1 + \psi_2 + \psi_3 \tag{VI-12}$$

These three simpler problems and their boundary conditions are thus:

Problem (1): $$\nabla^2 \psi_1 = 0 \tag{VI-13}$$

$$x = 0, \qquad \psi_1 = 0$$

$$x = x1, \qquad \psi_1 = \begin{cases} f_1(y) & 0 \le y < Y2 \\ 1 & y_2 \le y \le Y1 \end{cases}$$

$$y = 0, \qquad \psi_1 = 0$$

$$y = Y1, \qquad \psi_1 = 0$$

Problem (2): $$\nabla^2 \psi_2 = 0 \tag{VI-14}$$

$$x = 0, \qquad \psi_2 = \frac{y}{Y1}$$

$$x = x1, \qquad \psi_2 = 0$$

$$y = 0, \qquad \psi_2 = 0$$

$$y = Y1, \qquad \psi_2 = 0$$

Problem (3): $\nabla^2\psi_3 = 0$ (VI-15)

$$x = 0, \quad \psi_3 = 0$$

$$x = x1, \quad \psi_3 = 0$$

$$y = 0, \quad \psi_3 = 0$$

$$y = Y1, \quad \psi_3 = 1$$

When the above problems are solved and superposed one thus has the solution

$$\psi_{R1} = G_1\psi_1 + G_2\psi_2 + \ldots + G_{2i-1}\psi_{2i-1} + G_{2i}\,\psi_{2i} + \ldots + G_N\psi_N + G_{N+1} \qquad \text{(VI-16)}$$

where

$$G_1 = \sum_{k=1}^{\infty} \frac{2}{y1\,\sinh(\lambda_{1k}x1)} D_0^{1k} \sin(\lambda_{1k}\,y)\,\sinh(\lambda_{1k}x)$$

$$G_{2i-1} = \sum_{k=1}^{\infty} \frac{2}{y1\,\sinh(\lambda_{1k}x1)} (D_{2i-4}^{3k} + D_{2i-2}^{1k}) \sin(\lambda_{1k}y)\,\sinh(\lambda_{1k}x)$$

$$G_{2i} = \sum_{k=1}^{\infty} \frac{2}{y1\,\sinh(\lambda_{1k}x1)} D_{2i-2}^{2k} \sin(\lambda_{1k}y)\,\sinh(\lambda_{1k}x)$$

C-3

$$G_N = \sum_{k=1}^{\infty} \frac{2}{y1 \sinh(\lambda_{1k} x1)} D_{N-3}^{3k} \sin(\lambda_{1k} y) \sinh(\lambda_{1k} x)$$

$$G_{N+1} = \sum_{k=1}^{\infty} \frac{2}{y1 \sinh(\lambda_{1k} x1)} \frac{1}{\lambda_{1k}} [\cos(\lambda_{1k} y1) -$$

$$\cos(k\pi)] \sin(\lambda_{1k} y) \sinh(\lambda_{1k} x) +$$

$$\sum_{k=1}^{\infty} \frac{2\cos(k\pi)}{k\pi \tanh(\lambda_{1k} x1)} \sin(\lambda_{1k} y) [\sinh(\lambda_{1k} y) -$$

$$\tanh(\lambda_{1k} x1) \cosh(\lambda_{1k} x)] + \sum_{k=1}^{\infty} \frac{4 \sin^2(k\pi/2)}{k\pi \sinh(\mu_{1k} y1)}$$

$$\sin(\mu_{1k} x) \sinh(\mu_{1k} y)$$

and $\lambda_1 = \frac{k\pi}{y1}$, $\mu_1 = \frac{k\pi}{x1}$. The coefficient D's are

$$D_0^{1k} = A_{1k}$$

$$D_{N-3}^{3k} = A_{3k} \cos(N-3)\lambda_k h + B_{3k} \sin(N-3)\lambda_k h$$

$$D_n^{Jk} = A_{Jk} \cos(n-3)\lambda_k h + B_{Jk} \sin(n-3)\lambda_k h \qquad \text{(VI-17)}$$

where $J = 1, 2, 3,$ $\quad 1 < n < N$, and $\lambda_k = \lambda_{1k}$ for sub-region R1. The coefficients A's and B's are:

$$A_{1k} = \frac{1}{\lambda_k^3 h^2} \cos(2\lambda_k h) + \frac{1}{2\lambda_k^2 h} \sin(2\lambda_k h) + \frac{1}{\lambda_k} - \frac{1}{\lambda_k^3 h^2}$$

$$A_{2k} = -\frac{2}{\lambda_k^3 h^2}\cos(2\lambda_k h) - \frac{2}{2\lambda_k^2 h}\sin(2\lambda_k h) + \frac{2}{\lambda_k^3 h^2}$$

$$A_{3k} = \left(-\frac{1}{\lambda_k} + \frac{1}{\lambda_k^3 h^2}\right)\cos(2\lambda_k h) + \frac{3}{2\lambda_k^2 h}\sin(2\lambda_k h) - \frac{1}{\lambda_k^3 h^2}$$

$$B_{1k} = \frac{1}{2\lambda_k^2 h}\cos(2\lambda_k h) - \frac{1}{\lambda_k^3 h^2}\sin(2\lambda_k h) + \frac{3}{2\lambda_k^2 h}$$

$$B_{2k} = -\frac{2}{\lambda_k^2 h}\cos(2\lambda_k h) + \frac{2}{\lambda_k^3 h^2}\sin(2\lambda_k h) - \frac{2}{\lambda_k^2 h}$$

$$B_{3k} = \frac{3}{2\lambda_k^2 h}\cos(2\lambda_k h) + \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_k^3 h^2}\right)\sin(2\lambda_k h) + \frac{1}{2\lambda_k^2 h}$$

(VI-18)

Again, for subregion R1, $\lambda_k = \lambda_{1k}$. From the solution given above, the corresponding solution for the derivatives with respect to x and y (which is needed in evaluation of the heat flux) can be easily derived. For example, by differentiating the equation (VI-20) or (VI-21) with respect to y we have:

$$\frac{\partial\psi_{R1}}{\partial y} = G_{y1}\psi_1 + G_{y2}\psi_2 + \dots G_{y(2i-1)}\psi_{2i-1} + G_{y2i}\psi_{2i}$$

$$+ \dots + G_{yN}\psi_N + G_{y(N+1)} \qquad \text{(VI-19)}$$

where

$$G_{y1} = \sum_{k=1}^{\infty} \frac{2\lambda_{1k}}{y1 \sinh(\lambda_{1k} x1)} D_0^{1k} \cos(\lambda_{1k} y) \sinh(\lambda_{1k} x)$$

$$G_{y(2i-1)} = \sum_{k=1}^{\infty} \frac{2\lambda_{1k}}{y1 \sinh(\lambda_{1k} x1)} (D_{2i-4}^{3k} + D_{2i-2}^{1k})$$

$$\cos(\lambda_{1k} y) \sinh(\lambda_{1k} x)$$

$$G_{y2i} = \sum_{k=1}^{\infty} \frac{2\lambda_{1k}}{y1 \sinh(\lambda_{1k} x1)} D_{2i-2}^{2k} \cos(\lambda_{1k} y) \sinh(\lambda_{1k} x)$$

$$G_{yN} = \sum_{k=1}^{\infty} \frac{2\lambda_{1k}}{y1 \sinh(\lambda_{1k} x1)} D_{N-3}^{3k} \cos(\lambda_{1k} y) \sinh(\lambda_{1k} x)$$

$$G_{y(N+1)} = \sum_{k=1}^{\infty} \frac{2}{y1 \sinh(\lambda_{1k} x1)} [\cos(\lambda_{1k} y1) - \cos(k\pi)]$$

$$\cos(\lambda_{1k} y) \sinh(\lambda_{1k} x) + \sum_{k=1}^{\infty} \frac{2\lambda_{1k} \cos(k\pi)}{k\pi \tanh(\lambda_{1k} x1)}$$

$$\cos(\lambda_{1k} y)[\sinh(\lambda_{1k} x) - \tanh(\lambda_{1k} x1) \cosh(\lambda_{1k} x)]$$

$$+ \sum_{k=1}^{\infty} \frac{4\mu_{1k} \sin^2(k\pi/2)}{k\pi \sinh(\mu_{1k} y1)} \sin(\mu_{1k} x) \cosh(\mu_{1k} y)$$

where $\lambda_{1k} = \frac{k\pi}{y1}$, $\mu_{1k} = \frac{k\pi}{x1}$

VI.1.3 The FA Solution for Subregion R2

The same procedure used for subregion R1 may apply to solve the analytic solution ψ_{R2} for subregion R2, given in equation (VI-10). We thus have

$$\psi_{R2} = Q_1\psi_1 + \ldots + Q_{2i-1}\psi_{2i-1} + Q_{2i}\ \psi_{2i} + \ldots\ Q_N\psi_N +$$

$$Q'_1\ \psi'_1 + \ldots + Q'_{2i-1}\ \psi'_{2i-1} + Q'_{2i}\ \psi'_{2i} + \ldots$$

$$+ Q'_N\ \psi'_N + Q'_{N+1} \qquad \text{(VI-20)}$$

where

$$Q_1 = \sum_{k=1}^{\infty} - \frac{2}{y2\ \tanh(\lambda_{2k}x2)} D_0^{1k}\ \sin(\lambda_{2k}y)\ [\sinh(\lambda_{2k}x) -$$

$$\tanh(\lambda_{2k}x2)\ \cosh(\lambda_{2k}x)]$$

$$Q_{2i-1} = \sum_{k=1}^{\infty} - \frac{2}{y2\ \tanh(\lambda_{2k}x2)}\ (D_{2i-4}^{3k} + D_{2i-2}^{1k})\ \sin(\lambda_{2k}x)$$

$$-\tanh(\lambda_{2k}x2)\ \cosh(\lambda_{2k}x)]$$

$$Q_{2i} = \sum_{k=1}^{\infty} - \frac{2}{y2\ \tanh(\lambda_{2k}x2)} D_{2i-2}^{2k}\ \sin(\lambda_{2k}y)[\sinh(\lambda_{2k}x)$$

$$-\tanh(\lambda_{2k}x2)\ \cosh(\lambda_{2k}x)]$$

$$Q_N = \sum_{k=1}^{\infty} - \frac{2}{y2\ \tanh(\lambda_{2k}x2)} D_{N-3}^{3k}\ \sin(\lambda_{2k}y)[\sinh(\lambda_{2k}x)$$

$$-\tanh(\lambda_{2k}x2)\ \cosh(\lambda_{2k}x)]$$

$$Q'_1 = \sum_{k=1}^{\infty} \frac{2}{y2\ \sinh(\lambda_{2k}x2)} D_0^{1k}\ \ \sin(\lambda_{2k}y)\ \sinh(\lambda_{2k}x)$$

$$Q'_{2i-1} = \sum_{k=1}^{\infty} \frac{2}{y2 \sinh(\lambda_{2k} y2)} (D^{3k}_{2i-4} + D^{1k}_{2i-2}) \sin(\lambda_{2k} y)$$

$$\sinh(\lambda_{2k} x)$$

$$Q'_{2i} = \sum_{k=1}^{\infty} \frac{2}{y2 \sinh(\lambda_{2k} x2)} D^{2k}_{2i-2} \sin(\lambda_{2k} y) \sinh(\lambda_{2k} x)$$

$$Q'_{N} = \sum_{k=1}^{\infty} \frac{2}{y2 \sinh(\lambda_{2k} x2)} D^{3k}_{N-3} \sin(\lambda_{2k} y) \sinh(\lambda_{2k} x)$$

$$Q'_{N+1} = \sum_{k=1}^{\infty} \frac{4 \sin(k\pi/2)}{k\pi \sinh(\mu_{2k} y2)} \sin(\mu_{2k} x) \sinh(\mu_{2k} y)$$

The D's have the same definition as before (equations (VI-17) and (VI-18)), but $\lambda_k = \lambda_{2k}$ for subregion R2, $\lambda_{2k} = \frac{k\pi}{y2}$ and $\mu_{2k} = \frac{k\pi}{x2}$. Again the $\frac{\partial \psi_{R2}}{\partial y}$ and $\frac{\partial \psi_{R2}}{\partial x}$ can be obtained easily from the solution ψ_{R2} equation (VI-20).

$$\frac{\partial \psi_{R2}}{\partial y} = Q_{y1}\psi_1 + \cdots + Q_{y(2i-1)}\psi_{2i-1} + Q_{y2i}\psi_{2i} + \cdots +$$

$$Q_{yN}\psi_N + Q'_{y1}\,\psi'_1 + \cdots + Q'_{y(2i-1)}\psi'_{2i-1}$$

$$+ Q'_{y2i}\psi'_{2i} + \cdots + Q'_{yN}\psi'_N + Q'_{y(N+1)} \qquad \text{(VI-21)}$$

where

$$Q_{y1} = \sum_{k=1}^{\infty} - \frac{2\lambda_{2k}}{y2 \tanh(\lambda_{2k} x2)} D^{1k}_{0} \cos(\lambda_{2k} y) [\sinh(\lambda_{2k} x)$$

$$-\tanh(\lambda_{2k}x2)\ \cosh(\lambda_{2k}x)]$$

$$Q_{y(2i-1)} = \sum_{k=1}^{\infty} - \frac{2\lambda_{2k}}{y2\tanh(\lambda_{2k}x2)}(D^{3k}_{2i-4} + D^{1k}_{2i-2})\ \cos(\lambda_{2k}y)$$

$$[\sinh(\lambda_{2k}x) - \tanh(\lambda_{2k}x2)\ \cosh(\lambda_{2k}x)]$$

$$Q_{y2i} = \sum_{k=1}^{\infty} \frac{2\lambda_{2k}}{y2\tanh(\lambda_{2k}x2)}\ D^{2k}_{2i-2}\ \cos(\lambda_{2k}y)[\sinh(\lambda_{2k}x)$$

$$-\tanh(\lambda_{2k}x2)\ \cosh(\lambda_{2k}x)]$$

$$Q_{yN} = \sum_{k=1}^{\infty} \frac{2\lambda_{2k}}{y2\tanh(\lambda_{2k}x2)}\ D^{3k}_{N-3}\ \cos(\lambda_{2k}y)[\sinh(\lambda_{2k}x)$$

$$-\tanh(\lambda_{2k}x2)]$$

$$Q'_{y1} = \sum_{k=1}^{\infty} \frac{2\lambda_{2k}}{y2\sinh(\lambda_{2k}x2)}\ D^{1k}_{0}\cos(\lambda_{2k}y)\ \sinh(\lambda_{2k}x)$$

$$Q'_{y(2i-1)} = \sum_{k=1}^{\infty} \frac{2\lambda_{2k}}{y2\sinh(\lambda_{2k}x2)}\ (D^{3k}_{2i-4} + D^{1k}_{2i-2})\cos(\lambda_{2k}y)$$

$$\sinh(\lambda_{2k}y)$$

$$Q'_{y2i} = \sum_{k=1}^{\infty} \frac{2\lambda_{2k}}{y2\sinh(\lambda_{2k}x2)}\ D^{2k}_{2i-2}\ \cos(\lambda_{2k}y)\ \sinh(\lambda_{2k}x)$$

$$Q'_{yN} = \sum_{k=1}^{\infty} \frac{2\lambda_{2k}}{y2\sinh(\lambda_{2k}x2)}\ D^{3k}_{N-3}\ \cos(\lambda_{2k}y)\ \sinh(\lambda_{2k}x)$$

$$Q'_{y(N+1)} = \sum_{k=1}^{\infty} \frac{4\mu_{2k}\sin^2(k\pi/2)}{k\pi\sinh(\lambda_{2k}y1)}\sin(\mu_{2k}x)\cosh(\mu_{2k}y)$$

VI.1.4 The FA Solution to Subregion R3

The FA solution for subregion R3 (equation (VI-11)) by using the separation of variables, is:

$$\psi_{R3} = S_1\psi_1' + S_2\psi_2' + \ldots + S_{2i-1}\psi_{2i-1}' + S_{2i}\psi_{2i}' + \ldots + S_N\psi_N' + S_{N+1} \quad \text{(VI-22)}$$

where

$$S_1 = \sum_{k=1}^{\infty} - \frac{2}{y3\tan(\lambda_{3k}x3)} D_0^{1k} \sin(\lambda_{3k}y)[\sinh(\lambda_{3k}x) - \tanh(\lambda_{3k}x3)\cosh(\lambda_{3k}x)]$$

$$S_{2i-1} = \sum_{k=1}^{\infty} - \frac{2}{y3\tanh(\lambda_{3k}x3)} (D_{2i-4}^{3k} + D_{2i-2}^{1k}) \sin(\lambda_{3k}y) - \tanh(\lambda_{3k}x3)\cosh(\lambda_{3k}x)]$$

$$S_{2i} = \sum_{k=1}^{\infty} - \frac{2}{y3\tanh(\lambda_{3k}x3)} D_{2i-2}^{2k} \sin(\lambda_{3k}y)[\sinh(\lambda_{3k}x) - \tanh(\lambda_{3k}x3)\cosh(\lambda_{3k}x)]$$

$$S_N = \sum_{k=1}^{\infty} - \frac{2}{y3\tanh(\lambda_3x3)} D_{N-3}^{3k} \sin(\lambda_{3k}y)\ [\sinh(\lambda_{3k}x) - \tanh(\lambda_{3k}x3)\cosh(\lambda_{3k}x)]$$

$$S_{N+1} = \sum_{k=1}^{\infty} - \frac{2}{y3\tanh(\lambda_{3k}x3)} \cdot \frac{1}{\lambda_{3k}}[\cos(\lambda_{3k}y2) - \cos(k\pi)]$$

$$[\sinh(\lambda_{3k}x) - \tanh(\lambda_{3k}x3)\cosh(\lambda_{3k}x)] -$$

$$\sum_{k=1}^{\infty} \frac{2\cos(k\pi)}{k\pi\sinh(\lambda_{3k}x3)} \sin(\lambda_3 y)\sinh(\lambda_{3k}x) +$$

$$\sum_{k=1}^{\infty} \frac{4\sin^2(k\pi/2)}{k\sinh(\mu_{3k}x3)} \sin(\mu_{3k}x)\sinh(\mu_{3k}y)$$

And:

$$\frac{\partial\psi_{R3}}{\partial y} = S_{y1}\psi_1' + \ldots + S_{y(2i-1)}\psi_{2i-1}' + S_{y2i}\psi_{2i}' + \ldots +$$

$$S_{yN}\psi_N' + S_{y(N+1)} \qquad \text{(VI-23)}$$

where:

$$S_{y1} = \sum_{k=1}^{\infty} - \frac{2\lambda_{3k}}{y3\tanh(\lambda_{3k}x3)} D_0^{1k} \cos(\lambda_{3k}y)[\sinh(\lambda_{3k}x)$$

$$-\tanh(\lambda_{3k}x3)\cosh(\lambda_{3k}x)]$$

$$S_{y(2i-1)} = \sum_{k=1}^{\infty} - \frac{2\lambda_{3k}}{y3\tanh(\lambda_{3k}x3)} (D_{2i-4}^{3k} + D_{2i-2}^{1k})$$

$$\cos(\lambda_{3k}y)\,[\sinh(\lambda_{3k}y) - \tanh(\lambda_{3k}x3)$$

$$\cosh(\lambda_{3k}x)]$$

$$S_{y2i} = \sum_{k=1}^{\infty} - \frac{2\lambda_{3k}}{y3\tanh(\lambda_{3k}x3)} D_{2i-2}^{2k} \cos(\lambda_{3k}y) \ [\sinh(\lambda_{3k}x)$$

$$-\tanh(\lambda_{3k}x3) \cosh(\lambda_{3k}x)]$$

$$S_{yN} = \sum_{k=1}^{\infty} - \frac{2\lambda_{3k}}{y3\tanh(\lambda_{3k}x3)} D_{N-3}^{3k} \cos(\lambda_{3k}y)[\sinh(\lambda_{3k}x)$$

$$-\tanh(\lambda_{3k}x3) \cosh(\lambda_{3k}x)]$$

$$S_{y(N+1)} = \sum_{k=1}^{\infty} - \frac{2\lambda_{3k}}{y3\tanh(\lambda_{3k}x3)} (\cos(\lambda_{3k}y2) - \cos(k\pi)]$$

$$\cos(\lambda_{3k}y)[\sinh(\lambda_{3k}y) - \tanh(\lambda_{3k}x3) \cosh(\lambda_{3k}x)]$$

$$- \sum_{k=1}^{\infty} \frac{2\cos(k\pi)}{y3\sinh(\lambda_{3k}x3)} \cos(\lambda_{3k}y) \sinh(\lambda_{3k}x)$$

$$+ \sum_{k=1}^{\infty} \frac{4\sin^2(k\pi/2)}{x3\sinh(\mu_{3k}y3)} \sin(\mu_{3k}x) \cosh(\mu_{3k}y)$$

Again, the D's have the same expressions, as given in equations (VI-17) and (VI-18), for subregion R3 $\lambda_k = \lambda_{3k}$, $\lambda_{3k} = \frac{k\pi}{y3}$, and $\lambda_{3k} = \frac{k\pi}{x3}$.

As was mentioned, if the unknown temperatures, $\psi_2, \ldots \psi_N$ and $\psi_1', \psi_2', \ldots \psi_N'$ are predicted, then equations (VI-16) (VI-20) and (VI-21) will provide the solution for any point in the respective region.

VI.2 Steady Two Dimensional Heat Conduction with Groove

In industrial machines, it is often necessary to have a groove in the solid slab. For example, the oil reservoir, in bearing thermocouple, is good for temperature measurement. Instalation of such devices produces grooves in the pipes or channels, causing distortion in the temperature distribution and heat flux. A typical two-dimensional groove is shown in Figure (VI-1). In order to solve this problem with the FA method the entire region is subdivided into three subregions, R1, R2, and R3, as shown in Figure (VI-2). The analytic solution for each of these subregions was obtained in terms of unknown nodal point variables ψ_1, ψ_2,... ψ_N and ψ_1', ψ_2',...,ψ_N' which are equations (VI-16), (VI-20), and (VI-22). In this section the finite analytic solutions of the problem are obtained by combining the three analytic solutions in the subregions. In order to obtain the numerical results the iterative method described in Section (VI-1) is employed, the procedure of which is briefly outlined here. The following calculation steps for the finite analytic algorithm are also depicted in the flow chart given in Table (VI-1).

(a) Start with an initial guessed approximation of the temperatures, $\psi_n^{(0)}$ and $\psi_n'^{(0)}$, for all points (n) on the common boundaries.

(b) Find the temperatures TW's, TE's, TW''s and TE''s from the analytic solution in each subregion or equations (VI-16), (VI-20), and (VI-22) respectively (TE''s and TW''s are obtained from the same equation (VI-20)).

(c) Employ the 9-point FA formula (equation (VI-3)) to find the new ψ_n and ψ_n' as described in the equations (VI-4) and (VI-5).

(d) Repeat steps (b) and (c) until a convergence criterion is met.

(e) Once the temperatures $\psi_1, \psi_2, \cdots \psi_N$ and $\psi_1', \psi_2', \cdots \psi_N'$ are known the analytic solution for each subregion R1, R2, and R3 (equation (VI-16), (VI-20), and (VI-22)) may provide the solution at any desirable point in the whole region. The temperature gradient at any point is also available from equations (VI-19), (VI-21) and (VI-23). The numerical results for the steady two-dimensional heat conduction with three different sizes of groove, as shown in Figure (VI-1), is presented here.

In all three cases both isotherms and temperature, the gradient on the bottom surface of the slabs are plotted in Figures (VI-6), (VI-7), and (VI-8). In these figures the isotherms are plotted with a temperature interval ψ of

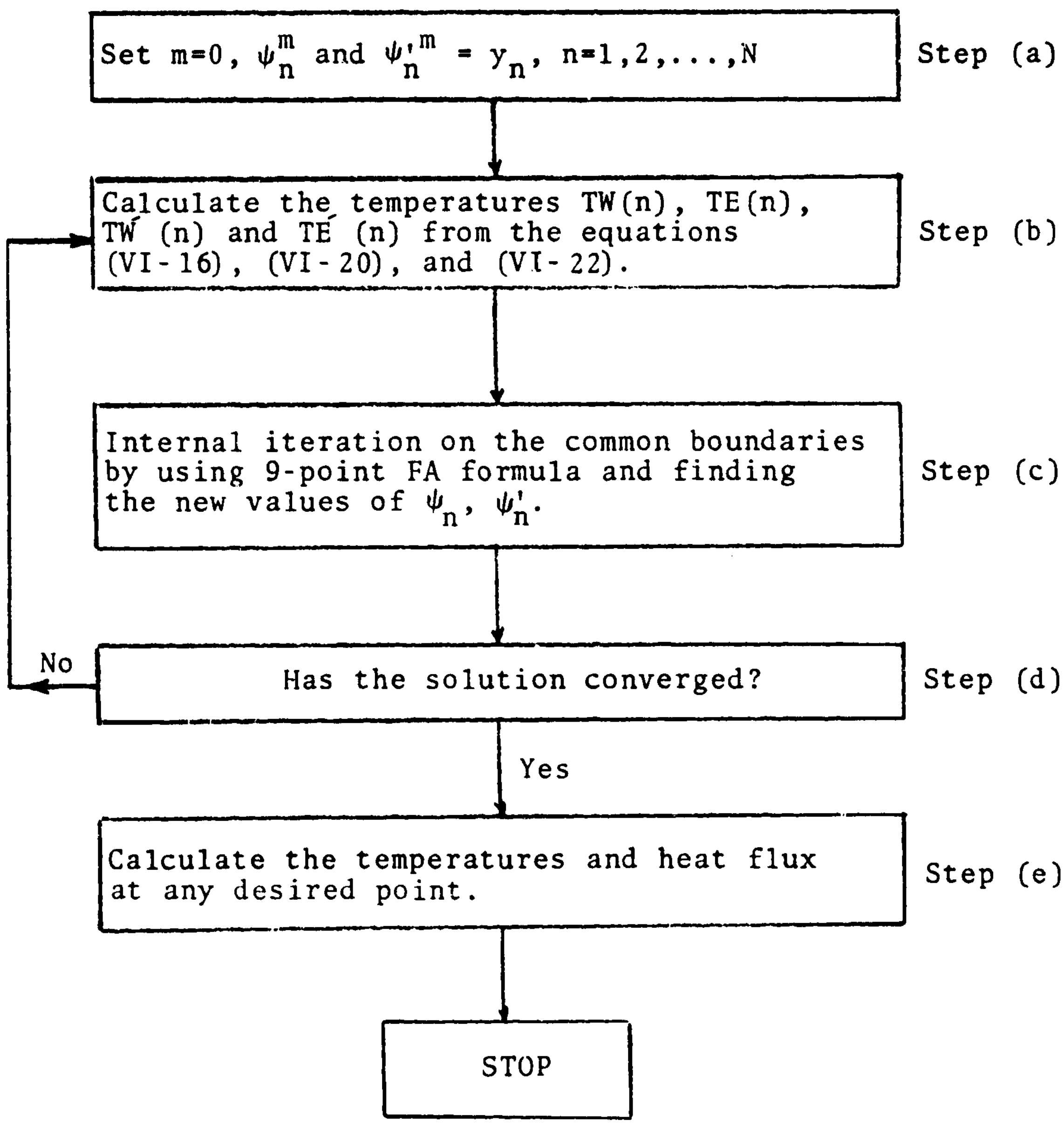


Table (VI-1). The FA Algorithm

.25. It should be noted that the number of nodes used to form the temperature function on each boundary is chosen to be five (N=5) in the present calculation.

VI.2.1 Numerical Results for Symmetric Groove

The solution to the problem with the groove between the symmetric slabs is shown in Figure (IV-6). In the calculation the geometry is taken as x1 = 3, Y1 = 1, x2 = .5, Y2 = .5, x3 = 3, and Y3 = 1. The temperature on the top, including the groove walls, is normalized to one and zero on the bottom surfaces while the temperature on the side walls of the slabs is assumed to vary linearly from zero at the bottom to one at the top. This is to simulate the temperature distribution at the large distance from the groove, where the heat conduction is essentially one-dimensional in the y direction and the temperature distribution is linear in y. If we take five nodal points on each common boundary, there will be a total of six unknown temperatures ψ_2, ψ_3, ψ_4, ψ_2', ψ_3', and ψ_4' in this finite analytic procedure because the temperature is at the bottom and top of each boundary $\psi_1 = \psi_1' = 0$ and $\psi_5 = \psi_5' = 1$ are known from the boundary conditions. The numerical results are obtained by successive iteratation of the nodal values on the common boundaries with the calculation procedure discussed in the previous sections. It should be noticed that the initial values for ψ_n and ψ_n'

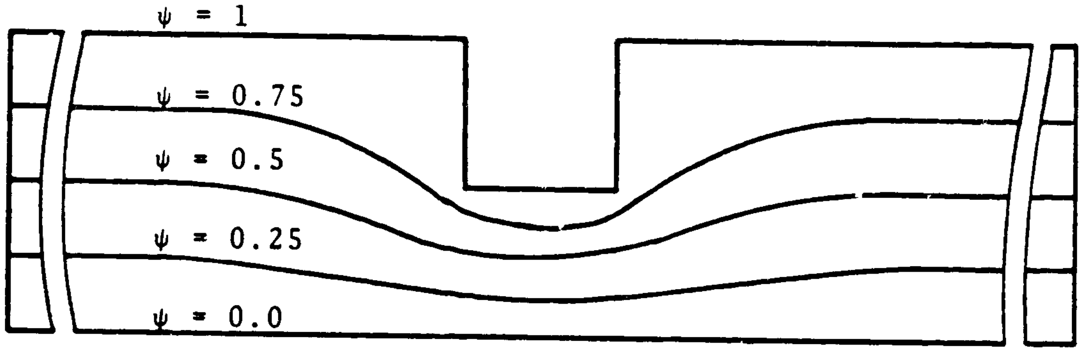
to begin the iteration are chosen as a linear function of y, that is $\psi_n^0 = \psi_n'^0 = y_n (n = 1,2,\ldots,N)$. The FA solution converges after twelve iterations within the error of 0(10^{-}[illegible]). The corresponding nodal values ψ_n, ψ_n' and also the temperatures TW(n), TE(n), TW$'$(n) and TE$'$(n) (n = 1,2,3,4,5) are presented in Table (VI-2). Since the temperatures at the common boundaries are known, the solution for the whole region can be calculated from the analytic solution given in equations (VI-16), (VI-20), and (VI-22). The temperature gradient on the bottom surface $(\frac{\partial\psi}{\partial y}|_0)$ can also be obtained from equations (VI-19), (VI-21) and (VI-23). The isotherms and temperature gradient on the bottom surface are shown in Figure (VI-6). It shows that the temperature and heat flux distributions for the square groove extend to about the groove's width.

VI.2.2 Numerical Results for Unsymmetric Groove

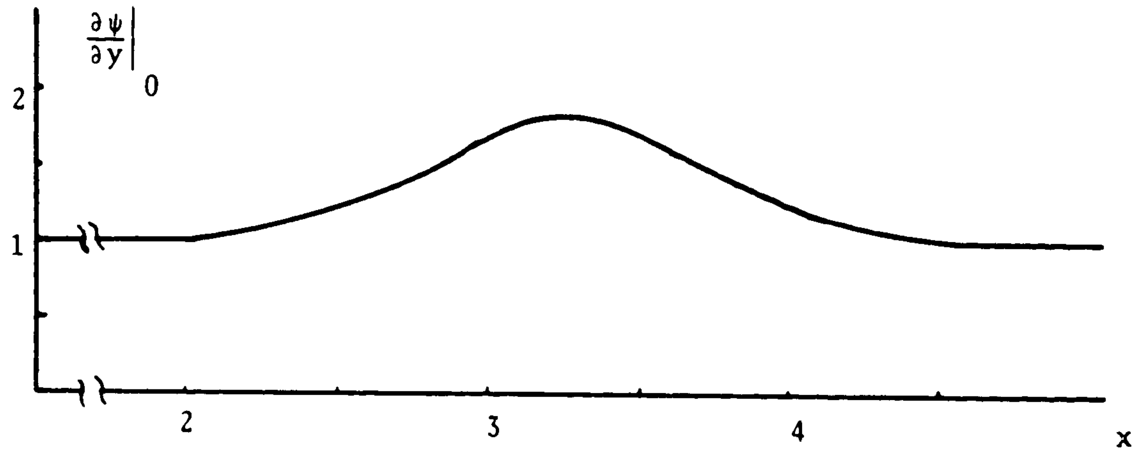
This problem is almost the same as the previous problem except the geometry is unsymmetric. The slab thickness on the right is set Y3 = .75 instead of Y3 = 1 and the other dimensions are kept the same as before. The numerical results of the FA solution for the temperatures on the common boundaries ψ_n and ψ_n' and TW(n), TE(n), TW$'$(n) and TE$'$(n) are listed in Table (VI-3) as well as some isotherms and temperature gradient for this problem are plotted in

n	TW(n)	ψ_n	TE(n)	$TW'(n)$	ψ'_n	$TE'(n)$
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.1974	0.2128	0.2263	0.2263	0.2128	0.1974
3	0.3965	0.4334	0.4618	0.4618	0.4334	0.3964
4	0.5965	0.6769	0.7182	0.7182	0.6768	0.5964
5	0.7719	1.0	1.0	1.0	1.0	0.7719

Table (VI-2). The Numerical Results for the Unknown Temperatures on the Common Boundaries and Temperatures TW(n), TE(n), $TW'(n)$, and $TE'(n)$ for Symmetric Groove.



(a)



(b)

Figure (VI-6). Isotherms Distortion and Temperature Gradient Distribution for Symmetric Groove.

(a) Distortion of Isotherms Near the Groove.
(b) Temperature Gradient Distribution on the Bottom Surface of the Slab.

n	TW(n)	ψ_n	TE(n)	TW′(n)	ψ'_n	TE′(n)
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.1976	0.2132	0.2271	0.2297	0.2192	0.2084
3	0.3967	0.4340	0.4630	0.4674	0.4448	0.4177
4	0.5967	0.6773	0.7191	0.7229	0.6890	0.6262
5	C.7720	1.0	1.0	1.0	1.0	0.8096

Table (IV-3). The Numerical Results for the Unknown Temperatures on the Common Boundaries and Temperatures TW(n), TE(n), TW′(n), and TE′(n) for Unsymmetric Groove.

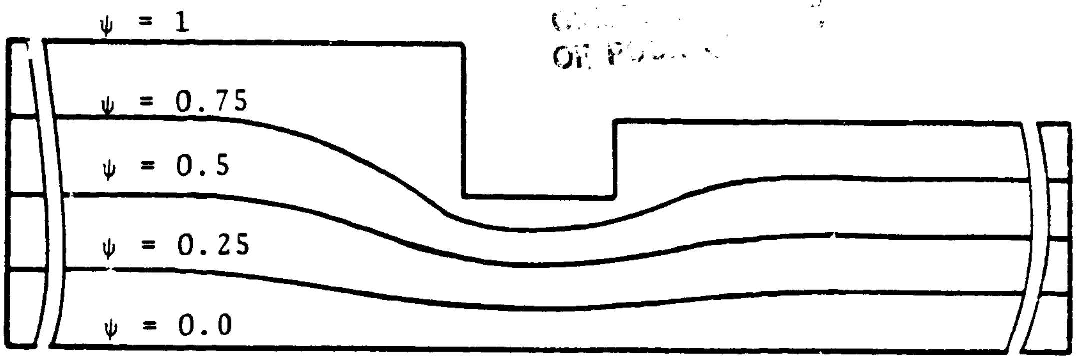
Figure (VI-7). It illustrates the maximum heat flux in the groove which has shifted to the right.

VI.2.3 Numerical Results for Step Groove

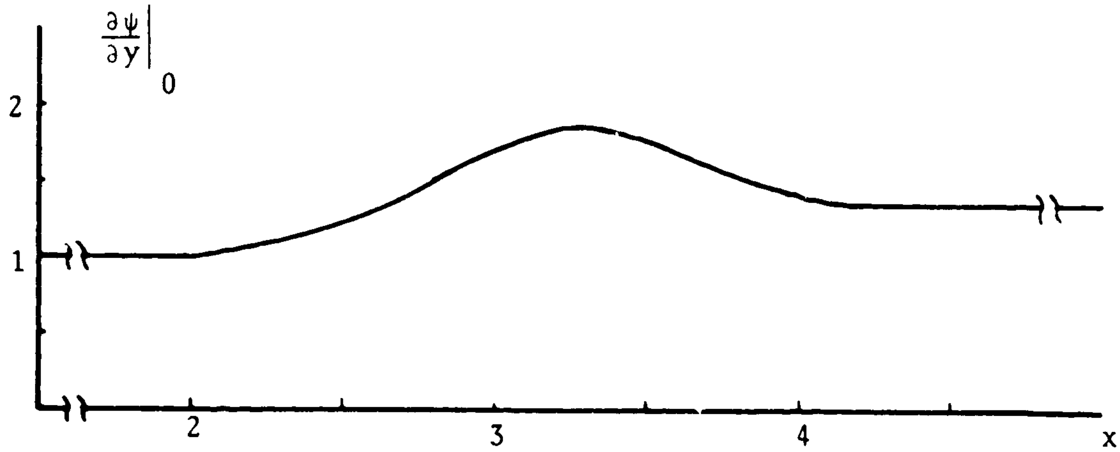
In this problem Y3 = .5 and the other dimensions are kept the same as before. The numerical results to this problem are shown in Table (VI-4). Some isotherms and a temperature gradient (on the bottom surface) are also shown in Figure (VI-8).

VI.3 Discussion

In this chapter a different procedure of the FA method was described. Although this solution procedure is demonstrated for the Laplace equation with simple boundary conditions, it can be extended to other linear partial differential equations with more complicated boundary conditions. In this new procedure for the FA solution the probelm was subdivided into only three subregions instead of subdividing it into many subregions as in the convention procedure of the FA method. This is only possible when the governing equation in each subregion can be solved analytically with corresponding boundary conditions. The FA analytic solution thus is continuous and differentiable in each subregion domain. The errors in this FA method are introduced only on the function used to approximate the common boundary



(a)



(b)

Figure (VI-7). Isotherms Distortion and Temprature Gradient Distribution for Unsymmetric Groove.

(a) Distortion of Isotherms Near the Groove.
(b) Temperature Gradient Distribution on the Bottom Surface of the slab.

n	TW(n)	ψ_n	TE(n)	TW′(n)	ψ'_n	TE′(n)
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.1983	0.2146	0.2302	0.2428	0.2429	0.2464
3	0.3978	0.4361	0.4676	0.4891	0.4883	0.4944
4	0.5976	0.6789	0.7225	0.7416	0.7389	0.7455
5	0.7724	1.0	1.0	1.0	1.0	1.0

Table (VI-4). The Numerical Results for the Unknown Temperatures on the Common Boundaries and Temperatures TW(n), TE(n), TW′(n), and TE′(n) for Step Groove.

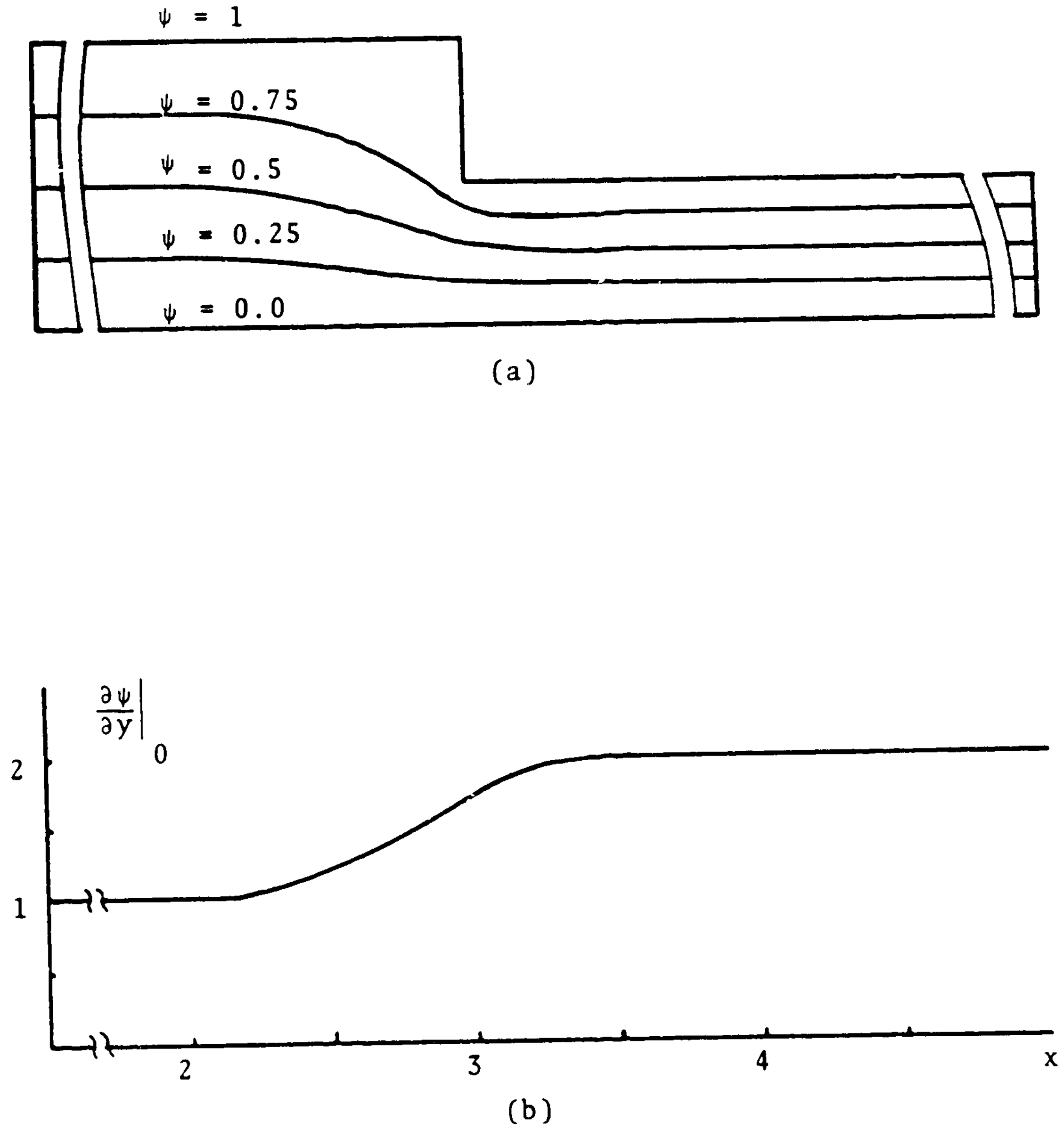


Figure (VI-8). Isotherms distortion and temperature Gradient Distribution for Step Groove.

(a) Distortion of Isotherms Near the Groove.
(b) Temperature Gradient Distribution on the Bottom Surface of the slab.

conditions. In this present study the piecewise continuous second-order polynomials are used to approximate the common boundary conditions, thus the FA method's error is of the order (h^3). The errors can be reduced if the common boundary functions are approximated by higher order piecewise continuous polynomials or third degree spine functions. It should also be noted that the present problem can be solved with more complicated boundary conditions on the top and bottom surfaces instead of one at the top and zero at the bottom. In such a case the FA solution can be obtained by the same procedure for equations (VI-16) and (VI-19) to (VI-23). The only change needed is that the N+1th coefficients for the equations (VI-16) and (VI-19) through (VI-23) are reevaluated for the new boundary conditions.

CHAPTER VII

CONCLUSION AND SUGGESTIONS

The finite analytic (FA) method introduced by Chen and Li [3] was applied to solve the Poisson equation numerically. The FA method utilizes the analytic solution obtained in a subregion of the problem to form the algebraic functional relation between a nodal value in the subregion with its neighboring nodal values. In the present investigation many FA formulas with different kinds of boundary conditions were derived. The accuracy of the FA method was examined for the case of the Poisson equation which represents a two-dimensional heat conduction in rectangular shape with uniform heat generation having two insulated boundaries and two isothermal boundaries. In this case, the FA solution was compared with the 5-point central finite difference (FD) solution and the finite element (FE) solution and also with the exact solution. The FA solution was shown to be more accurate than the other methods under the same overall conditions.

Another new solution procedure utilizing the FA method was applied to solve the Laplace equation with complex geometry. In this new procedure, instead of subdividing the problem into the regular small subregions, the problem was subdivided into regions where the common boundary conditions are approximated by a piecewise polynomial that the analytic solution can be obtained. In the example considered, three large subregions were considered and the 9-point FA formula is only used for the boundary nodal points.

The finite analytic (FA) solution,although requiring more analytic manipulation,involves the following advantages:

1. The computational time for the finite analytic solution is not a problem for the linear partial differential equation because the finite analytic coefficients are invariant and can be calculated once for each subregion with the same type of boundary conditions.

2. The accuracy of the FA solution, although depending on the grid size, is less sensitive to it than the FD solution. Indeed, the only approximation made in the FA method is that the boundary functions f_E, f_S, f_W, and f_N are approximated by second-degree polynomials.

3. The algebraic equation system, derived from the FA methods is stable and has faster convergence rates.

4. The FA solution is differentiable so the derivative of the dependent variable obtained from the FA method is generally more reliable.

Regarding the suggestion for the further use of the finite analytic method it should be remarked that the application of the finite analytic method is not limited to the partial differential equations of heat transfer problems. The FA method is a general numerical solution technique for problems involving either ordinary differential equations or partial differential ones. The principle of the FA method may be readily extended to the three-dimensional problems. In the case of steady three-dimensional heat conduction problems, the local subregion may be a rectangular cube and the finite analytic formula (similar to equation (VI-4)) may be derived. The FA method is especially powerful in solving the governing equations with linear partial differential operator of constant coefficient because the subregion may be taken relatively large. In case of nonlinear partial differential equations, normally the local linearization is made to obtain the local analytic solution. In this case, the FA method requires the local linearization and the approximations of the boundary conditions. However, the FA method eliminates the error in using difference approximations due to the Taylor series expansion of the derivatives as with the finite difference. The FA method minimizes the problem of the numerical diffusion that happens to the upwinding approximation used in the finite difference or finite element methods when the coefficients to the lower derivative terms are large (i.e., large Reynolds number flow in fluid mechanics problems). The accuracy of the FA solution may be improved by using higher degree polynomials on the boundaries. In this case a 17-point FA formula can be derived with five nodal points on each boundary so that the boundary functions are approximated by a 4th degree polynomial.

Further details of numerical treatments and analytic solution techniques used in the present investigation refer to references [18] through [25].

APPENDIX A

THE FA SOLUTION OF TWO DIMENSIONAL HEAT CONDUCTION

```
C    ******************************************
C    APLIED FINITE ANALYTIC METHOD
C    TWO DIMENTIONAL STE  ADY STATE HEAT CONDUC-
C    TION WITH CONSTANT HEAT GENERATION.LEFT
C    AND LOWER SIDES ARE ADIABATIC.RIGHT AND
C    UPER SIDES ARE AT 0 DEGREE.
C    A GAUSS-SEIDEL ITERATION IS EMPLOYED.
C    ******************************************
    PROGRAM FDT(INPUT,OUTPUT,RESULT,TAPE5=INPUT,
+   TAPE6=OUTPUT,TAPE8=RESULT)
    DIMENSION AA(9),AB(9),AC(9),AD(9),AE(9),
+   AF(9),AG(9),AH(9),AI(9),T(10,10)
    READ*DX,DY,N,ITMAX,EPS
C   N IS NUMBER OF DX ON A SIDE
    G=DX/DY
C
C
C              T   +T   =-1
C               XX   YY
C
    CALL COEF 1(DX,DY,G,AA)
    PRINT (8,301)
    PRINT (8,302),(AA(I),I=1,9)
    CALL COEF 2(DX,DY,G,AB,AC)
    PRINT (8,303)
    PRINT (8,304),(AB(I),AC(I),I=1,9)
    CALL COEF 3(DX,DY,G,AE,AF)
    PRINT (8,305)
    PRINT (8,304),(AE(I),AF(I),I=1,9)
    CALL COEF 4(DX,DY,G,AD,AG,AH,AI)
    PRINT (8,306)
    PRINT (8,304),(AD(I),AG(I),I=1,9)
    PRINT (8,307)
    PRINT (8,304),(AH(I),AI(I),I=1,9)
C
C    ******************************************************
C    ALL THE COEFFICIENTS NOW ARE KNOWN .
C    LET US SOLVE THE SYSTEM OF EQUATIONS TO GET THE
C    NUMERICAL SOLUTION.
C    ******************************************************
C
    NP1=N+1
    DO 33 I=1,NP1
    T(I,NP1)=0.0
    DO 33 J=1,NP1
    T(NP1,J)=0.0
    DO 33 JJ=2,NP1
 33 T(I,J)=0.0
    ITER=0
111  ITER=ITER+1
    DO 240 I=1,N
    DO 240 J=1,N
    HOLDT=T(I,J)
    IF(I.EQ.1.AND.J.EQ.1) GO TO 200
    IF(I.EQ.1.AND.J.EQ.2) GO TO 300
    IF(I.EQ.1)            GO TO 400
    IF(I.EQ.2.AND.J.EQ.2) GO TO 600
    IF(I.EQ.2.AND.J.EQ.1) GO TO 500
    IF(I.EQ.2)            GO TO 700
    IF(J.EQ.1)            GO TO 800
    IF(J.EQ.2)            GO TO 900
```

```
          GO TO 112
    200   T(1,1)=AI(1)*T(3,3)+AI(2)*T(3,2)+AI(3)*T(3,1)+AI(7)
         +*T(1,3)+AI(8)*T(2,3)+AI(9)
          GO TO 240
    300   T(1,2)=AG(1)*T(3,3)+AG(2)*T(3,2)+AG(3)*T(3,1)+AG(7)
         +*T(1,3)+AG(8)*T(2,3)+AG(9)
          GO TO 240
    400   T(1,J)=AC(1)*T(3,J+1)+AC(2)*T(3,J)+AC(3)*T(3,J-1)+
         +AC(4)*T(2,J-1)+AC(5)*T(1,J-1)+AC(7)*T(1,J+1)+AC(8)*
         +T(2,J+1)+AC(9)
          GO TO 240
    500   T(2,1)=AH(1)*T(3,3)+AH(2)*T(3,2)+AH(3)*T(3,1)+
         +AH(7)*T(1,3)+AH(8)*T(2,3)+AH(9)
          GO TO 240
    600   T(2,2)=AD(1)*T(3,3)+AD(2)*T(3,2)+AD(3)*T(3,1)+
         +AD(7)*T(1,3)+AD(8)*T(2,3)+AD(9)
          GO TO 240
    700   T(2,J)=AB(1)*T(3,J+1)+AB(2)*T(3,J)+AB(3)*T(3,J-1)
         ++AB(4)*T(2,J-1)+AB(5)*T(1,J-1)+AB(7)*T(1,J+1)+AB(8)
         +*T(2,J+1)+AB(9)
          GO TO 240
    800   T(I,1)=AF(1)*T(I+1,3)+AF(2)*T(I+1,2)+AF(3)*T(I+1,1)
         ++AF(5)*T(I-1,1)+AF(6)*T(I-1,2)+AF(7)*T(I-1,3)+AF(8)*
         +T(I,3)+AF(9)
          GO TO 240
    900   T(I,2)=AE(1)*T(I+1,3)+AE(2)*T(I+1,2)+AE(3)*T(I+1,1)
         ++AE(5)*T(I-1,1)+AE(6)*T(I-1,2)+AE(7)*T(I-1,3)
         ++AE(8)*T(I,3)+AE(9)
          GO TO 240
    112   T(I,J)=AA(1)*T(I+1,J+1)+AA(2)*T(I+1,J)+AA(3)*T(I+1,
         +J-1)+AA(4)*T(I,J-1)+AA(5)*T(I-1,J-1)+AA(6)*T(I-1,J)+
         +AA(7)*T(I-1,J+1)+AA(8)*T(I,J+1)+AA(9)
    240   CONTINUE
          IF(ABS(T(I,J)-HOLDT).LE.EPS) GO TO 118
          IF(ITER-ITMAX) 111,111,118
C         END OF THE ITERATION.
C         PRINT THE RESULT.
    118   PRINT(8,398)
          PRINT (8,399),N,ITER
          PRINT (8,401)
          DO 119 II=1,NP1
          I=NP1+1-II
    119   PRINT (8,999),(T(I,J),J=1,NP1)
    999   FORMAT(3X,5(3X,E10.5)/)
    301   FORMAT(////3X,*THE COEFFICIENTS OF THE POINTS WHICH*/
         +3X,*SOUROUD POINT P FOR ELEMENT A *//15X,*AA(I)*/)
    302   FORMAT(3X,*C(NE)*,3X,E14.8/3X,*C(N) *,3X,E14.8/
         +3X,*C(NW)*,3X,E14.8/3X,*C(W) *,3X,E14.8/3X,
         +*C(SW)*,3X,E14.8/3X,*C(S) *,3X,E14.8/3X,*C(SE)*,3X,
         +E14.8/3X,*C(E) *,3X,E14.8/3X,*C(K) *,3X,E14.8/)
    303   FORMAT(//3X,*FOR ELEMENT BC, AB(I) AND AC(I) ARE*/
         +3X,*THE COEFFICIENTS OF THE POINTS WHICH SOURROUND*/
         +3X,*POINTS P AND Q,RESPECTIVELY.*//15X,*AB(I)*,12X,
         +*AC(I)*/)
    304   FORMAT(3X,*C(NE)*,2(3X,E14.8)/3X,*C(N) *,2(3X,E14.8)
         +/3X,*C(NW)*,2(3X,E14.8)/3X,*C(W) *,2(3X,E14.8)/3X,
         +*C(SW)*,2(3X,E14.8)/3X,*C(S) *,2(3X,E14.8)/3X,*C(SE)*,
         +2(3X,E14.8)/3X,*C(E) *,2(3X,E14.8)/3X,*C(K) *,2(3X,E14.8)/)
    305   FORMAT(//3X,*FOR ELEMENT EF, AE(I) AND AF(I) ARE*/
         +3X,*THE COEFFICIENTS OF THE POINTS WHICH SOURROUND*/
         +3X,*THE POINTS P AND Q(ON SIDE X=0),RESPECTIVELY.*//
```

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```
+      15X,*AE(I)*,12X,*AF(I)*/)
  306  FORMAT(//3X,*FOR ELEMENT DGHI, AD(I),AG(I),AH(I)*/
+      3X,*AND AI(I) ARE THE COEFFICIENTS OF THE POINTS*/
+      3X,*WHICH SOURROUND P AND Q.*//15X,*AD(I)*,12X,*AG(I)*/)
  307  FORMAT(/15X,*AH(I)*,12X,*AI(I)*/)
  398  FORMAT(3X,*STEADY-STATE HEAT CONDUCTION WITH*
+      /3X,*CONSTANT HEAT GENERATION IN A FLAT PLATE WITH*/
+      3X,*PARAMETERS,*/)
  399  FORMAT(3X,*H        =*,I3/3X,*ITMAX =*,I3/)
  401  FORMAT(3X,*THE TEMPRATURE FIELD IS GIVEN BY*/)
       END
C
C
C      ****************************************************
C          FIND THE COEFFICIENTS FOR ELEMENT A
C      A(I) ARE THE COEFFICIENTS OF THE POINTS WHICH
C      SURROUND POINT P.
C      ****************************************************
       SUBROUTINE COEF 1(DX,DY,G,AA)
       DIMENSION AA(9)
       PI=3.141592654
C
C        NW(3)          N(2)           NE(1)
C          :---------------------------:
C          :              :            :
C          :              P:           :
C      W(4):--------------*------------:E(8)
C          :              :            :
C          :              :            :
C          :______________:____________:
C        SW(5)          S(6)           SE(7)
C
C
       DO 10 I=1,9
   10   AA(I)=0.0
       DO 20 I=1,9
       DO 20 J=1,25
       B=(PI/2.)*J
       K=2*J-1
       X=(PI/2.)*K
       GO TO(1,2,1,2,1,2,1,2,3),I
    1   AAA=(1./B-2./B**3)*(SINH(B*G)/SINH(2.*B*G)+SINH(B/G)
+      /SINH(2.*B/G))*SIN(B)
       GO TO 20
    2  AAA=4./B**3*SIN(B)*SINH(B*G)/SINH(2.*B*G)
       GO TO 20
    3   AAA=2.*DX**2/B**3*SINH(B/G)*SIN(B)*(1./TANH
+      (2.*B/G)-1./SINH(2.*B/G)+1./SINH(B/G)-1./TANH
+      (B/G))
   20    AA(I)=AA(I)+AAA
       RETURN
       END
C
C
C
C****************************************************
C          FIND THE COEFFICIENTS FOR ELEMENT BC,
C      AB(I) AND AC(I) ARE COEFFICIENTS OF THE POINTS
C      WHICH SURROUND POINTS P AND Q RESPECTIVELY.
C****************************************************
C
```

```
      SUBROUTINE COEF 2(DX,DY,G,AB,AC)
      DIMENSION AB(9),AC(9)
      PI=3.141592654
C
C           NW                 N                  NE
C
C              ---------------------------------
C              :                 :               :
C              :                 :               :
C             W:-----------------*---------------:E
C              :                 : P             :
C              :                 :   Q           :
C              :_________________*_______________:
C           SW/////////////////////////////////////SE
C                               DY/DY=0
C
      DO 10 I=1,9
      AB(I)=0.0
   10   AC(I)=0.0
      DO 20 I=1,9
      DO 20 J=1,25
      B=(PI/2.)*J
      K=2*J-1
      X=(PI/2.)*K
      GO TO(1,2,3,4,5,6,7,8,9),I
    1  ABB=(2./X**2+(2./X-8./X**3)*SIN(X))*SINH(X*G/2.)/SIN
     +H(X*G)*COS(X/2.)+(1./B-2./B**3)*SIN(B)*COSH(B/G)/COSH(
     +2.*B/G)
      ACC=(2./X**2+(2./X-8./X**3)*SIN(X))*SINH(X*G/2.)/SIN
     +H(X*G)+(1./B-2./B**3)*SIN(B)/COSH(2.*B/G)
      GO TO 15
    2 ABB=4./B**3*SIN(B)*COSH(B/G)/COSH(2.*B/G)
      ACC=4./B**3*SIN(B)/COSH(2.*B*G)
      GO TO 15
    3  ABB=(2./X**2+(2./X-8./X**3)*SIN(X))*(1.-TANH(X*G/2.)
     +/TANH(X*G))*COSH(X*G/2.)*COS(X/2)+(1./B-2./B**3)*SIN(B
     +)*COSH(B/G)/COSH(2.*B/G)
      ACC=(2./X**2+(2./X-8./X**3)*SIN(X))*(COSH(X*G/2.)-SI
     +NH(X*G/2.)/TANH(X*G))+(1./B-2./B**3)*SIN(B)/COSH(2.*B/
     +G)
      GO TO 15
    4  ACC=(-8./X**2+16./X**3*SIN(X))*(1.-TANH(X*G/2.)/TANH
     +(X*G))*COSH(X*G/2.)
      ABB=ACC*COS(X/2.)
      GO TO 15
    5  ACC=(6./X**2-8./X**3*SIN(X))*(1.-TANH(X*G/2.)/TANH(X*
     +G))*COSH(X*G/2.)
      ABB=ACC*COS(X/2.)
      GO TO 15
    6  ABB=0.0
      ACC=0.0
      GO TO 15
    7 ABB=(6./X**2-8./X**3*SIN(X))*SINH(X*G/2.)/SINH(X*G)
     +*COS(X/2.)
      ACC=(6./X**2-8./X**3*SIN(X))*SINH(X*G/2.)/SINH(X*G)
      GO TO 15
    8  ACC=(-8./X**2+16./X**3*SIN(X))*SINH(X*G/2.)/SINH(X*
     +G)
      ABB=ACC*COS(X/2.)
      GO TO 15
    9 ABB=2.*DX**2/B**3*(1.-COSH(B/G)/COSH(2.*B*G))*SIN(B)
```

```
      ACC=2.*DX**2/B**3*(1.-1./COSH(2.*B*G))*SIN(B)
   15 AB(I)=AB(I)+ABB
   20 AC(I)=AC(I)+ACC
      RETURN
      END
C     ***************************************************
C        FIND THE COEFFICIENTS FOR ELEMENT EF,
C      AE(I) AND AP(I) ARE THE COEFFICIENTS CF POINTS
C      WHICH SURROUND POINTS P AND Q RESPECTIVELY.
C     ***************************************************
      SUBROUTINE COEF 3(DX,DY,G,AE,AP)
      DIMENSION AE(9),AP(9)
      PI=3.141592654
C
C
C              NW              N              NE
C               :-----------------------------:
C              /:              :              :
C              /:              :              :
C             W/*--------------*--------------:E
C DT/DX=0      /: Q            : P            :
C              /:              :              :
C              /:______________:______________:
C              SW              S              SE
C
C
      DO 10 I=1,9
      AE(I)=0.0
   10 AP(I)=0.0
      DO 20 I=1,9
      DO 20 J=1,25
      B=(PI/2.)*J
      K=2*J-1
      X=(PI/2.)*K
      GO TO(1,2,3,4,5,6,7,8,9),I
    1 AEE=(2./X**2+(2./X-8./X**3)*SIN(X))*SINH(X*G/2.)/SIN
     +H(X*G)*COS(X/2.)+(1./B-2./B**3)*SIN(B)*COSH(B/G)/COSH(
     +2.*B*G)
      APP=(2./X**2+(2./X-8./X**3)*SIN(X))*SINH(X*G/2.)/SIN
     +H(X*G)+(1./B-2./B**3)*SIN(B)/COSH(2.*B*G)
      GO TO 16
    2 APP=(-8./X**2+16./X**3*SIN(X))*SINH(X*G/2.)/SINH(X*G)
      AEE=APP*COS(X/2.)
      GO TO 16
    3 APP=(6./X**2-8./X**3*SIN(X))*SINH(X*G/2.)/SINH(X*G)
      AEE=APP*COS(X/2.)
      GO TO 16
    4 AEE=0.0
      APP=0.0
      GO TO 16
    5 APP=(6./X**2-8./X**3*SIN(X))*(1.-TANH(X*G/2.)/TANH
     +(X*G))*COSH(X*G/2.)
      AEE=APP*COS(X/2.)
      GO TO 16
    6 APP=(-8./X**2+16./X**3*SIN(X))*(1.-TANH(X*G/2.)/TAN
     +H(X*G))*COSH(X*G/2.)
      AEE=APP*COS(X/2.)
      GO TO 16
    7 AEE=(2./X**2+(2./X-8./X**3)*SIN(X))*(1.-TANH(X*G/2.
     +)/TANH(X*G))*COSH(X*G/2.)*COS(X/2.)+(1./B-2./B**3)*SIN
     +(B)*COSH(B*G)/COSH(2.*B*G)
```

```
      AFF=(2./X**2+(2./X-8./X**3)*SIN(X))*(1.-TANH(X*G/2.
     +)/TANH(X*G))*COSH(X*G/2.)+(1./B-2./B**3)*SIN(B)/COSH(
     +2.*B*G)
      GO TO 16
    8 AFF=4./B**3*SIN(B)/COSH(2.*B*G)
      AEE=AFF*COSH(B*G)
      GO TO 16
    9 AEE=2.*DY**2/B**3*SIN(B)*(1.-COSH(B/G)/COSH(2.*B/G))
      AFF=2.*DY**2/B**3*SIN(B)*(1.-1./COSH(2.*B*G))
   16 AE(I)=AE(I)+AEE
   20 AF(I)=AF(I)+AFF
      RETURN
      END
C     ***************************************************
C        FIND THE COEFFICIENTS FOR ELEMENT EGHI
C      AD(I),AG(I),AH(I) AND AI(I) ARE THE COEFFICIENTS
C      OF POINTS WHICH SURROUND POINTS P,Q1,Q2 AND Q3
C      RESPECTIVELY.
C     ***************************************************
      SUBROUTINE COEF 4(DX,DY,G,AD,AG,AH,AI)
      DIMENSION AD(9),AG(9),AH(9),AI(9)
      PI=3.141592654
C
C
C               NW                   N                   NE
C                :-------------------:-------------------:
C               /:                   :                   :
C               /:                   :                   :
C DT/DX=0      W/*-------------------*-------------------:E
C               /: Q2                : P                 :
C               /: Q3                : Q1                :
C               /*-------------------*-------------------
C              SW/////////////////////S///////////////////SE
C                               DT/DY=0
C
C
      DO 10 I=1,9
      AD(I)=0.0
      AG(I)=0.0
      AH(I)=0.0
   10 AI(I)=0.0
      DO 20 I=1,9
      DO 20 J=1,25
      B=(PI/2.)*J
      K=2*J-1
      X=(PI/2.)*K
      GO TO(1,2,3,4,5,6,7,8,9),I
    1 ADD=(2./X**2+(2./X-8./X**3)*SIN(X))*(COSH(X*G/2.)/CO
     +SH(X*G)*COSH(X/(2.*G))/COSH(X/G))*COS(X/2.)
      AGG=(2./X**2+(2./X-8./X**3)*SIN(X))*(COSH(X*G/2.)/CO
     +SH(X*G)*COS(X/2.)/COSH(X/G))
      AHH=(2./X**2+(2./X-8./X**3)*SIN(X))*(COS(X/2.)/COSH(
     +X*G)*COSH(X/(2.*G))/COSH(X/G))
      AII=(2./X**2+(2./X-8./X**3)*SIN(X))*(1./COSH(X*G)+1.
     +/COSH(X/G))
      GO TO 17
    2 ADD=(-8./X**2+16./X**3*SIN(X))*COSH(X/(2.*G))/COS
     +H(X/G)*COS(X/2.)
      AGG=(-8./X**2+16./X**3*SIN(X))*COS(X/2.)/COSH(X/G)
      AHH=(-8./X**2+16./X**3*SIN(X))*COSH(X/(2.*G))/COS
     +H(X/G)
```

```
      AII=(-8./X**2+16./X**3*SIN(X))/COSH(X/G)
      GO TO 17
    3 AII=(6./X**2-8./X**3*SIN(X))/COSH(X/G)
      AHH=  AII*COSH(X/(2.*G))
      AGG=  AII*COS(X/2.)
      ADD=  AHH*COS(X/2.)
      GO TO 17
    4 ADD=0.0
      AGG=0.0
      AHH=0.0
      AII=0.0
      GO TO 17
    5 ADD=0.0
      AGG=0.0
      AHH=0.0
      AII=0.0
      GO TO 17
    6 ADD=0.0
      AGG=0.0
      AHH=0.0
      AII=0.0
      GO TO 17
    7 AII=(6./X**2-8./X**3*SIN(X))/COSH(X*G)
      AGG=  AII*COSH(X*G/2.)
      AHH=AII*COS(X/2.)
      ADD=AGG*COS(X/2.)
      GO TO 17
    8 AII=(-8./X**2+16./X**3*SIN(X))/COSH(X*G)
      AGG=AII*COSH(X*G/2.)
      AHH=AII*COS(X/2.)
      ADD=AGG*COS(X/2.)
      GO TO 17
    9 ADD=8.*DX**2/X**3*SIN(X)*(1.-COSH(X/(2.*G))/COSH
     (X/G))*COS(X/2.)
      AII=8.*DX**2/X**3*SIN(X)*(1.-1./COSH(X/G))
      AGG=AII*COS(X/2.)
      AHH=8.*DX**2/X**3*SIN(X)*(1.-COSH(X/(2.*G))/COSH
     (X/G))
   17 AD(I)=AD(I)+ADD
      AG(I)=AG(I)+AGG
      AH(I)=AH(I)+AHH
   20 AI(I)=AI(I)+AII
      RETURN
      END
```

APPENDIX B

THE FA SOLUTION FOR LAPLACE EQUATION WITH COMPLEX GEOMETRY

```
C     ************************************************
C     FINITE ANALYTIC SOLUTION OF THE LAPLACE
C     FOR COMPLEX GEOMETRY OF HEAT TRANSFER OR HEAT
C     POTENTIAL FLOW WITH STEP OR CONTRUCTION.
C     ************************************************
      PROGRAM FTC(INPUT,OUTPUT,OUHEAT,TAPE5=INPUT
     +,TAPE6=OUTPUT,TAPE8=OUHEAT)
      DIMENSION FIOLD(25),FINEW(25),FIOLDP(25),FINEWP(25
     +),TA(25),TB(25),TC(25),TD(25),HOLD(25),HOLDP(25)
      ITER=0
C     ************************************************
C                    INPUT DATAS
C      X1,Y1,X2,Y2,X3 AND Y3 ARE DIMENSIONS OF THE PROBLEM.
C      N IS THE NUMBER OF POINTS AND THIS NUMBER HAS TO BE
C      ODD NUMBER. ITMAX IS THE MAXIMUM NUMBER OF ITERATION
C      . IK IS THE NUMBER OF FOURIER'S SERIES TERMS AND
C      IK SHOULD BE MORE THAN 100.
C      EPS IS A CONTROL FOR CONVERGENCY.
C     ************************************************
      READ*X1,Y1,X2,Y2,X3,Y3,N,ITMAX,IK,EPS
      PRINT 998
      PRINT 997,X1,Y1,X2,Y2,X3,Y3,N,ITMAX,IK,EPS
  997 FORMAT(//3X,*X1=*,F4.2,3X,*Y1=*,F4.2/3X,
     +*X2=*,F4.2,3X,*Y2=*,F4.2/3X,*X3=*,F4.2,3X,*Y3=*
     +,F4.2/3X,*N = *,I3,3X,*ITMAX=*,I3/3X,*IK =*,I4,
     +3X,*EPS=*,F10.6//)
  998 FORMAT(///3X,*INPUT DATAS ARE FOLLOING *//)
      NM1=N-1
      FLOATN=N-1
      DY=Y2/FLOATN
      XP1=X1-DY
      XP2=X2-DY
      DO 101 I=1,N
      FLOATI=I-1
C      ************************************************
C       FIRST GUESS SHOULD BE MADE TO START THE
C       ITERATION. FIOLD(I),FIOLDP(I) ARE THE
C       TEMPRATURE ON THE LINE BETWEEN THE PART 1
C       ,2 AND PART 2,3 RESPECTIVELY.
C      ************************************************
      FIOLD(I)=1./Y2*FLOATI*DY
  101 FIOLDP(I)=FIOLD(I)
  105 ITER=ITER+1
      DO 704 I=2,NM1
      HOLD(I)=FIOLD(I)
  704 HOLDP(I)=FIOLDP(I)
      TA(1)=0.0
      TB(1)=0.0
      TC(1)=0.0
      TD(1)=0.0
      TB(N)=1.0
      TC(N)=1.0
      DO 104 I=2,NM1
      FLOATI=I-1
      YI=FLOATI*DY
C      ************************************************
C       THE ANALYTIC SOLUTION FOR PART 2 AND THEN
C       FIND THE TEMPRATURE AT CERTAIN POINTS (X=DY,
C       Y=DY,2DY,.....(N-1)DY).
C      ************************************************
      CALL TIB(5,5,IK,DY,N,YI,DY,FIOLD,FIOLDP,X2,Y2,TEB
```

```
+       ,DELX,DELY)
C       ************************************************
C         FIND THE TEMPRATURE AT CERTAIN POINTS IN
C         PART 2 ,(X=X2-DY,Y=DY,2DY,.......(N-1)DY).
C       ************************************************
          CALL TTB(5,5,IK,DY,N,YI,XP2,FIOLD,FIOLDP,X2,Y2,TCC
+         ,DELX,DELY)
          TB(I)=TBB
   104  TC(I)=TCC
        DO 102 I=2,N
        FLOATI=I-1
        YI=FLOATI*DY
C       ************************************************
C        THE ANALYTIC SOLUTION FOR PART 1 AND FIND
C        THE TEMPRATURE AT CERTAIN POINTS (X=X1-DY,
C        Y=DY,2DY,.....(N-1)DY).
C       ************************************************
        CALL TTA(5,5,IK,DY,N,YI,XP1,FIOLD,X1,Y1,Y2,TAA
+       ,DELX,DELY)
C       ************************************************
C        THE ANALYTIC SOLUTION FOR PART 3 AND FIND
C        THE TEMPRATURE AT CERTAIN POINTS (X=DY,Y=DY,
C        2DY,......,(N-1)DY).
C       ************************************************
        CALL TTD(5,5,IK,DY,N,YI,DY,FIOLDP,X3,Y3,Y2,TDD
+       ,DELX,DELY)
        TA(I)=TAA
   102  TD(I)=TDD
        DO 103 I=2,NM1
C       **************************************************
C        USING THE NINE-POINT FINITE DIFFERENTIAL FORMULA
C        TO CONTINUE THE ITERATION.
C       **************************************************
        FINEW(I)=(TA(I+1)+TA(I-1)+TB(I-1)+TB(I+1))*.0446854
+       13+(TA(I)+FIOLD(I+1)+TB(I)+FIOLD(I-1))*0.203531459
        FINEWP(I)=(TC(I+1)+TC(I-1)+TD(I+1)+TD(I-1))*0.0446854
+       13+(TC(I)+FIOLDP(I+1)+TD(I)+FIOLDP(I-1))*0.20351459
        FIOLD(I)=FINEW(I)
   103  FIOLDP(I)=FINEWP(I)
        DO 705 I=2,NM1
   705  IF(ABS(HOLD(I)-FIOLD(I)).LT.EPS.AND.ABS(HOLDP(I)
+       -FIOLDP(I)).LT.EPS) GO TO 706
        IF(ITER.LT.ITMAX) GO TO 105
   706  PRINT 200,ITER
   200  FORMAT(3X,*ITER=*,I4///)
        PRINT 201
   201  FORMAT(3X,*THE DISTRIBUTION OF THE TEMPRATURE*,
+       * ON THE LINE*/3X,*BETWEEN PART 1 AND PART 2 IS*/)
        PRINT 202,(FIOLD(I),I=1,N)
        PRINT 203
   203  FORMAT(3X,*THE DISTRIBUTION OF THE TEMPRATURE*
+       ,*ON THE LINE*/3X,*BETWEEN PART 2 AND 3 IS*//)
        PRINT 202,(FIOLDP(I),I=1,N)
   202  FORMAT(3X,11(F10.6)//)
C       **************************************************
C        THE CALCULATION IS DONE. NOW YOU WILL BE ASKED
C        ABOUT YOUR DISIRABLE POINTS THAT YOU WISH TO
C        HAVE IT'S TEMPRATURE AND HEAT CONDUCTION IN
C        X AND Y DIRECTION.
C        IX AND IY ARE TWO NUMBER,IF YOU WANT DT/DX AND
C        DT/DY JUST PUT IX=1,IY=1,OTHERWISE PUT SOME
```

```
C      OTHER NUMBER.
C      ****************************************************
  348 READ*X,Y,IX,IY,IK
       PRINT 350,X,Y
       X123=X1+X2+X3
       X12=X1+X2
       IF(X.EQ.X1) GO TO 369
       IF(X.LT.X1) CALL TTA(IX,IY,IK,DY,N,Y,X,FIOLD,X1,Y1
+      ,Y2,T,DELX,DELY)
       IF(X.GT.X1.AND.X.LT.X12) GO TO 345
       IF(X.GT.X12.AND.X.LT.X123) GO TO 346
       IF(X.EQ.X123) GO TO 346
       IF(X.EQ.X12) GO TO 345
  369 CALL TTA(IX,IY,IK,DY,N,Y,X,FIOLD,X1,Y1,Y2,T,DELX,DELY)
       GO TO 347
  345  X=X-X1
       CALL TTB(IX,IY,IK,DY,N,Y,X,FIOLD,FIOLDP,X2,Y2,T,
+      DELX,DELY)
       GO TO 347
  346  X=X-X12
       CALL TTD(IX,IY,IK,DY,N,Y,X,FIOLDP,X3,Y3,Y2,T,DELX
+      ,DELY)
  350  FORMAT(3X,*THE TEMPRATURE AND HEAT CON. AT*,
+      * POINT X=*,F5.2,* AND Y=*,F5.2,* ARE*)
  347  PRINT 349,T,DELX,DELY
  349  FORMAT(//3X,*T=*,F10.6,3X,*DT/DX=*,F10.6
+      ,3X,*DT/DY=*,F10.6)
       GO TO 348
       END
C      ***************************************************
C       ANALYTIC SOLUTION FOR PART 1.
C       ***************************************************
C                   _______________
C                   I              I
C                   I              I
C                   I              I
C             Y/Y1  I PART 1     * *
C                   I            * *
C                   I            * *
C                   I            * * F(Y)
C                   I____________*_*
C                          0
C
       SUBROUTINE TTA(IX,IY,IK,DY,N,Y,X,F,X1,Y1,Y2,TAA
+      ,DELX,DELY)
       DIMENSION C(25),F(25),CB(25),CD(25)
        TAA=0.0
        DELX=0.0
        DELY=0.0
       NP1=N+1
       DO 10 I=1,NP1
       CB(I)=0.0
       CD(I)=0.0
   10  C(I)=0.0
       PI=3.141592654
       DO 11 J=1,NP1
       DO 11 K=1,IK
       XL1=(PI/Y1)*K
       IF(J.EQ.1) GO TO 2
       IF(J.EQ.N) GO TO 3
       IF(J.EQ.NP1) GO TO 4
```

```
      JJ=J/2*2-J
      IF(JJ.EQ.0) GO TO 5
      A1=FUNA1(DY,XL1)
      B1=FUNB1(DY,XL1)
      A3=FUNA3(DY,XL1)
      B3=FUNB3(DY,XL1)
      D1=A1*COS((J-1)*XL1*DY)+B1*SIN((J-1)*XL1*DY)
      D3=A3*COS((J-3)*XL1*DY)+B3*SIN((J-3)*XL1*DY)
      D13=D1+D3
      CC=FUNCC(K,Y1,XL1,X1,D13,X,Y)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=FUNCBX(K,Y1,XL1,X1,D13,X,Y)
      IF(IY.EQ.1) CDD=FUNCBY(K,Y1,XL1,X1,D13,X,Y)
      GO TO 1
    2  A1=FUNA1(DY,XL1)
     D1=A1
      CC=FUNCC(K,Y1,XL1,X1,D1,X,Y)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=FUNCBX(K,Y1,XL1,X1,D1,X,Y)
      IF(IY.EQ.1) CDD=FUNCBY(K,Y1,XL1,X1,D1,X,Y)
      GO TO 1
    3  A3=FUNA3(DY,XL1)
     B3=FUNB3(DY,XL1)
     D3=A3*COS((J-3)*XL1*DY)+B3*SIN((J-3)*XL1*DY)
      CC=FUNCC(K,Y1,XL1,X1,D3,X,Y)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=FUNCBX(K,Y1,XL1,X1,D3,X,Y)
      IF(IY.EQ.1) CDD=FUNCBY(K,Y1,XL1,X1,D3,X,Y)
      GO TO 1
    4  CAC=XL1*(X+X1)
      IF(CAC.GT.630) CAC=630
      EZ1=EXP(-CAC)
      CAP=2.*XL1*X1
      IF(CAP.GT.630) CAP=630
      EZ2=EXP(-CAP)
      XLP1=PI/X1*K
      CA=XL1*X
      DAD=XL1*(X1-X)
      IF(DAD.GT.630) DAD=630
      CAA=2.*XLP1*Y1
      CAB=XLP1*(Y+Y1)
      CAD=XLP1*(Y1-Y)
      IF(CAA.GT.630) CAA=630
      IF(CAB.GT.630) CAB=630
      IF(CAD.GT.630) CAD=630
      IF(CA.GT.450) CA=450
      CC=2./Y1*(-1./XL1*COS(K*PI)+1./XL1*COS(XL1*Y2))*SIN(
     +XL1*Y)*(EXP(-DAD)-EZ1)/(1.-EZ2)+2.*COS(K*PI)/
     +(K*PI*TANH(XL1*X1))*SIN(XL1*Y)*(SINH(CA)-TANH(XL1*X1
     +)*COSH(CA))+4.*(SIN(PI/2.*K))**2/(K*PI)*(EXP(-CAD)
     +-EXP(-CAB))/(1.-EXP(-CAA))*SIN(XLP1*X)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=2./Y1*(-COS(K*PI)+COS(XL1*Y2))*
     +SIN(XL1*Y)*(EXP(-DAD)+EZ1)/(1.-EZ2)+2.*COS(K*PI)/
     +(K*PI*TANH(XL1*X1))*SIN(XL1*Y)*(COSH(CA)-TANH(XL1
     +*X1)*SINH(CA))*XL1+4.*(SIN(PI/2.*K))**2/(K*PI)*
     +(EXP(-CAD)-EXP(-CAB))/(1.-EXP(-CAA))*COS(XLP1*X)*
```

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```
     +   XLP1
         IF(IY.EQ.1) CDD=2./Y1*(-COS(K*PI)+COS(XL1*Y2))*
     +   COS(XL1*Y)*(EXP(-CAD)-EZ1)/(1.-EZ2)+2.*COS(K*PI)
     +   /(K*PI*TANH(XL1*X1))*COS(XL1*Y)*XL1*(SINH(CA)-
     +   TANH(XL1*X1)*COSH(CA))+4.*(SIN(PI/2.*K))**2/(K*PI)
     +   *(EXP(-CAD)+EXP(-CAB))/(1.-EXP(-CAA))*XLP1
     +   *SIN(XLP1*X)
         GO TO 1
    5   A2=FUNA2(DY,XL1)
        B2=FUNB2(DY,XL1)
        D2=A2*COS((J-2)*XL1*DY)+B2*SIN((J-2)*XL1*DY)
         CC=FUNCC(K,Y1,XL1,X1,D2,X,Y)
         IF(IX.GT.1) CBB=0.0
         IF(IY.GT.1) CDD=0.0
         IF(IX.EQ.1) CBB=FUNCBX(K,Y1,XL1,X1,D2,X,Y)
         IF(IY.EQ.1) CDD=FUNCBY(K,Y1,XL1,X1,D2,X,Y)
     1   C(J)=C(J)+CC
         CB(J)=CB(J)+CBB
         CD(J)=CD(J)+CDD
    11   CONTINUE
         DO 74 I=1,N
         DBT=CB(I)*F(I)
         DDT=CD(I)*F(I)
         DELX=DELX+DBT
         DELY=DELY+DDT
         TT=C(I)*F(I)
    74   TAA=TAA+TT
         TAA=TAA+C(NP1)
         DELX=DELX+CB(NP1)
         DELY=DELY+CD(NP1)
         RETURN
         END
C        **************************************************
C         ANALYTIC SOLUTION FOR PART 2.
C        **************************************************
C
C
C                               T=1
C                     *-+---------+-*
C                     * +         + *
C                     * +         + *
C            F1(Y)    * +         + *  F2(Y)
C                     * +         + *
C                     * +         + *
C                     *-+---------+-*
C                           T=0
C                          PART 2
C
C
C
         SUBROUTINE TTB(IX,IY,IK,DY,N,Y,X,FD,FP,X2,Y2,T
     +   ,DELX,DELY)
         DIMENSION C(25),F(25),FD(25),FP(25),CB(25),CD(25)
     +   ,CH(25),CG(25)
         NP1=N+1
         T=0.0
         DELX=0.0
         DELY=0.0
         PI=3.141592654
         DO 10 I=1,NP1
         F(I)=0.0
```

```
      CB(I)=0.0
      CD(I)=0.0
      CE(I)=0.0
      CG(I)=0.0
   10 C(I)=0.0
      DO 1 J=1,NP1
      DO 1 K=1,IK
      XL2=(PI/Y2)*K
      IF(J.EQ.1) GO TO 2
      IF(J.EQ.N) GO TO 3
      IF(J.EQ.NP1) GO TO 4
      JJ=J/2*2-J
      IF(JJ.EQ.0) GO TO 5
      A1=FUNA1(DY,XL2)
      B1=FUNB1(DY,XL2)
      A3=FUNA3(DY,XL2)
      B3=FUNB3(DY,XL2)
      D1=A1*COS((J-1)*XL2*DY)+B1*SIN((J-1)*XL2*DY)
      D3=A3*COS((J-3)*XL2*DY)+B3*SIN((J-3)*XL2*DY)
      D13=D1+D3
      CC=FUNDD(K,Y2,XL2,X2,D13,X,Y)
      PP=FUNCC(K,Y2,XL2,X2,D13,X,Y)
      IF(IX.EQ.1) CCC=FUNDEX(K,Y2,XL2,X2,D13,X,Y)
      IF(IY.EQ.1) PPP=FUNDEY(K,Y2,XL2,X2,D13,X,Y)
      IF(IX.EQ.1) CCCC=FUNCEX(K,Y2,XL2,X2,D13,X,Y)
      IF(IY.EQ.1) PPPP=FUNCEY(K,Y2,XL2,X2,D13,X,Y)
      IF(IX.GT.1) CCC=0.0
      IF(IX.GT.1) CCCC=0.0
      IF(IY.GT.1) PPP=0.0
      IF(IY.GT.1) PPPP=0.0
      GO TO 6
    2 D1=FUNA1(DY,XL2)
      CC=FUNDD(K,Y2,XL2,X2,D1,X,Y)
      PP=FUNCC(K,Y2,XL2,X2,D1,X,Y)
      IF(IX.EQ.1) CCC=FUNDEX(K,Y2,XL2,X2,D1,X,Y)
      IF(IY.EQ.1) PPP=FUNDEY(K,Y2,XL2,X2,D1,X,Y)
      IF(IX.EQ.1) CCCC=FUNCEX(K,Y2,XL2,X2,D1,X,Y)
      IF(IY.EQ.1) PPPP=FUNCEY(K,Y2,XL2,X2,D1,X,Y)
      IF(IX.GT.1) CCC=0.0
      IF(IY.GT.1) PPP=0.0
      IF(IX.GT.1) CCCC=0.0
      IF(IY.GT.1) PPPP=0.0
      GO TO 6
    3 A3=FUNA3(DY,XL2)
      B3=FUNB3(DY,XL2)
      D3=A3*COS((J-3)*XL2*DY)+B3*SIN((J-3)*XL2*DY)
      CC=FUNDD(K,Y2,XL2,X2,D3,X,Y)
      PP=FUNCC(K,Y2,XL2,X2,D3,X,Y)
      IF(IX.EQ.1) CCC=FUNDEX(K,Y2,XL2,X2,D3,X,Y)
      IF(IY.EQ.1) PPP=FUNDEY(K,Y2,XL2,X2,D3,X,Y)
      IF(IX.EQ.1) CCCC=FUNCEX(K,Y2,XL2,X2,D3,X,Y)
      IF(IY.EQ.1) PPPP=FUNCEY(K,Y2,XL2,X2,D3,X,Y)
      IF(IX.GT.1) CCC=0.0
      IF(IY.GT.1) PPP=0.0
      IF(IX.GT.1) CCCC=0.0
      IF(IY.GT.1) PPPP=0.0
      GO TO 6
    4 XLP2=PI/X2*K
      CAA=2.*XLP2*Y2
      CAB=XLP2*(Y+Y2)
      CAC=XLP2*(Y2-Y)
```

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      IF(CAA.GT.630) CAA=630
      IF(CAB.GT.630) CAB=630
      IF(CAD.GT.630) CAD=630
      CC=4.*(SIN(K*PI/2.))**2/(K*PI)*SIN(XLP2*X)*
     +(EXP(-CAD)-EXP(-CAB))/(1.-EXP(-CAA))
      PP=0.0
      IF(IX.EQ.1) CCC=4.*(SIN(PI/2.*K))**2/(K*PI)*XLP2
     +*COS(XLP2*X)*(EXP(-CAD)-EXP(-CAB))/(1.-EXP(-CAA))
      IF(IX.GT.1) CCC=0.0
      CCCC=0.0
      PPPP=0.0
      IF(IY.EQ.1) PPP=4.*(SIN(PI/2.*K))**2/(K*PI)*XLP2
     +*(EXP(-CAD)+EXP(-CAB))/(1.-EXP(-CAA))*SIN(XLP2*X)
      IF(IY.GT.1) PPP=0.0
      GO TO 6
    5 A2=FUNA2(DY,XL2)
      B2=FUNB2(DY,XL2)
      D2=A2*COS((J-2)*XL2*DY)+B2*SIN((J-2)*XL2*DY)
      CC=FUNDD(K,Y2,XL2,X2,D2,X,Y)
      PP=FUNCC(K,Y2,XL2,X2,D2,X,Y)
      IF(IX.EQ.1) CCC=FUNDEX(K,Y2,XL2,X2,D2,X,Y)
      IF(IY.EQ.1) PPP=FUNDEY(K,Y2,XL2,X2,D2,X,Y)
      IF(IX.EQ.1) CCCC=FUNCEX(K,Y2,XL2,X2,D2,X,Y)
      IF(IY.EQ.1) PPPP=FUNCEY(K,Y2,XL2,X2,D2,X,Y)
      IF(IX.GT.1) CCC=0.0
      IF(IY.GT.1) PPP=0.0
      IF(IX.GT.1) CCCC=0.0
      IF(IY.GT.1) PPPP=0.0
    6 P(J)=P(J)+PP
      CB(J)=CB(J)+CCC
      CD(J)=CD(J)+CCCC
      CH(J)=CH(J)+PPP
      CG(J)=CG(J)+PPPP
    1 C(J)=C(J)+CC
      DO 7 J=1,N
      CBDT=CB(J)*FD(J)+CD(J)*FP(J)
      CHGT=CH(J)*FD(J)+CG(J)*FP(J)
      DELX=DELX+CBDT
      DELY=DELY+CHGT
      TT=C(J)*FD(J)+P(J)*FP(J)
    7 T=T+TT
      T=T+C(NP1)
      DELX=DELX+CB(NP1)
      DELY=DELY+CH(NP1)
      RETURN
      END
C     **************************************************
C      ANALYTIC SOLUTION FOR PART 3.
C     **************************************************
      SUBROUTINE TTC(IX,IY,IK,DY,N,Y,X,F,X3,Y3,Y2,T
     +,DELX,DELY)
      DIMENSION C(25),F(25),CB(25),CD(25)
      NP1=N+1
      T=0.0
      DELX=0.0
      DELY=0.0
      PI=3.141592654
      DO 10 I=1,NP1
      CB(I)=0.0
      CD(I)=0.0
   10 C(I)=0.0
```

```
      DO 11 J=1,NP1
      DO 11 K=1,IK
      XL3=(PI/Y3)*K
      IF(J.EQ.1) GO TO 2
      IF(J.EQ.N) GO TO 3
      IF(J.EQ.NP1) GO TO 4
      JJ=J/2*2-J
      IF(JJ.EQ.0) GO TO 5
      A1=FUNA1(DY,XL3)
      B1=FUNB1(DY,XL3)
      A3=FUNA3(DY,XL3)
      B3=FUNB3(DY,XL3)
      D1=A1*COS((J-1)*XL3*DY)+B1*SIN((J-1)*XL3*DY)
      D3=A3*COS((J-3)*XL3*DY)+B3*SIN((J-3)*XL3*DY)
      D13=D1+D3
      CC=FUNDD(K,Y3,XL3,X3,D13,X,Y)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=FUNDEX(K,Y3,XL3,X3,D13,X,Y)
      IF(IY.EQ.1) CDD=FUNDEY(K,Y3,XL3,X3,D13,X,Y)
      GO TO 1
    2 D1=FUNA1(DY,XL3)
      CC=FUNDD(K,Y3,XL3,X3,D1,X,Y)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=FUNDEX(K,Y3,XL3,X3,D1,X,Y)
      IF(IY.EQ.1) CDD=FUNDEY(K,Y3,XL3,X3,D1,X,Y)
      GO TO 1
    3 A3=FUNA3(DY,XL3)
      B3=FUNB3(DY,XL3)
      D3=A3*COS((J-3)*XL3*DY)+B3*SIN((J-3)*XL3*DY)
      CC=FUNDD(K,Y3,XL3,X3,D3,X,Y)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=FUNDEX(K,Y3,XL3,X3,D3,X,Y)
      IF(IY.EQ.1) CDD=FUNDEY(K,Y3,XL3,X3,D3,X,Y)
      GO TO 1
    4 CAC=XL3*(X+X3)
      IF(CAC.GT.630) CAC=630
      CAP=XL3*X3*2.0
      IF(CAP.GT.630) CAP=630
      EZ1=EXP(-CAC)
      EZ2=EXP(-CAP)
      XLP3=PI/X3*K
      DAD=XL3*(X3-X)
      IF(DAD.GT.630) DAD=630
      CAD=XLP3*(Y3-Y)
      CAB=XLP3*(Y3+Y)
      CAA=2.*XLP3*Y3
      IF(CAD.GT.630) CAD=630
      IF(CAB.GT.630) CAB=630
      IF(CAA.GT.630) CAA=630
      CA=XL3*X
      IF(CA.GT.450) CA=450
      CC=-2./(Y3*TANH(XL3*X3))*(1./XL3*COS(XL3*Y2)-1./XL
     +3*COS(K*PI))*SIN(XL3*Y)*(SINH(CA)-TANH(XL3*X3)*
     +COSH(CA))
     +-2.*COS(K*PI)/(K*PI)*SIN(XL3*Y)*(EXP(-DAD)
     +-EZ1)/(1.-EZ2)+4.*(SIN(PI/2.*K))**2*SIN(XLP3
     +*X)/(K*PI)*(EXP(-CAD)-EXP(-CAB))/(1.-EXP(-CAA))
      IF(IX.GT.1) CBB=0.0
```

```
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=-2./(Y3*TANH(XL3*X3))*(COS(XL3*Y2)
     -COS(K*PI))*SIN(XL3*Y)*(COSH(CA)-TANH(XL3*X3)*SINH
     (CA))-2.*COS(K*PI)/(K*PI)*SIN(XL3*Y)*(EXP(-CAD)+E
     Z1)/(1.-EZ2)*XL3+4.*(SIN(PI/2.*K))**2*COS(XLP3*X)*
     XLP3/(K*PI)*(EXP(-CAD)-EXP(-CAB))/(1.-EXP(-CAA))
      IF(IY.EQ.1) CDD=-2./(Y3*TANH(XL3*X3))*(COS(XL3*Y2)-
     COS(K*PI))*COS(XL3*Y)*(SINH(CA)-TANH(XL3*X3)*COSH(CA)
     )-2.*COS(PI*K)/(K*PI)*COS(XL3*Y)*XL3*(EXP(-CAD)-EZ1)/
     (1.-EZ2)+4.*(SIN(PI/2.*K))**2*SIN(XLP3*X)/(PI*K)*XLP3
     *(EXP(-CAD)+EXP(-CAB))/(1.-EXP(-CAA))
      GO TO 1
    5 A2=FUNA2(DY,XL3)
      B2=FUNB2(DY,XL3)
      D2=A2*COS((J-2)*XL3*DY)+B2*SIN((J-2)*XL3*DY)
      CC=FUNCC(K,Y3,XL3,X3,D2,X,Y)
      IF(IX.GT.1) CBB=0.0
      IF(IY.GT.1) CDD=0.0
      IF(IX.EQ.1) CBB=FUNDEX(K,Y3,XL3,X3,D2,X,Y)
      IF(IY.EQ.1) CDD=FUNDEY(K,Y3,XL3,X3,D2,X,Y)
    1 CB(J)=CB(J)+CBB
      CD(J)=CD(J)+CDD
   11 C(J)=C(J)+CC
      DO 7 J=1,N
      DET=CB(J)*F(J)
      DDT=CD(J)*F(J)
      DELX=DELX+DET
      DELY=DELY+DDT
      TT=C(J)*F(J)
    7 T=T+TT
      T=T+C(NP1)
      DELX=DELX+CB(NP1)
      DELY=DELY+CD(NP1)
      RETURN
      END
C     ************************************************
C     ALL FUNCTIONS ARE LISTED BELOW.
C     ************************************************
      FUNCTION FUNA1(Y,X)
      FUNA1=COS(2.*X*Y)/(X**3*Y**2)+(SIN(2.*X*Y))/(2.*Y*X**2)
     +1./X-1./(X**3*Y**2)
      RETURN
      END
      FUNCTION FUNA2(Y,X)
      FUNA2=-2.*(COS(2.*X*Y))/(X**3*Y**2)-2.*(SIN(2.*X*Y))/(X**2
     *Y)+2./(X**3*Y**2)
      RETURN
      END
      FUNCTION FUNA3(Y,X)
      FUNA3=(-1./X+1./(X**3*Y**2))*COS(2.*X*Y)+1.5/(X**2*Y)
     *SIN(2.*X*Y)-1./(X**3*Y**2)
      RETURN
      END
      FUNCTION FUNB1(Y,X)
      FUNB1=.5/(Y*X**2)*COS(2.*X*Y)-1./(Y**2*X**3)*SIN(2.*X*Y)
     +1.5/(Y*X**2)
      RETURN
      END
      FUNCTION FUNB2(Y,X)
      FUNB2=-2./(Y*X**2)*COS(2.*X*Y)+2./(Y**2*X**3)*SIN(2.*X*Y
     )-2./(Y*X**2)
```

```
      RETURN
      END
      FUNCTION FUNB3(Y,X)
      FUNB3=1.5/(Y*X**2)*COS(2.*X*Y)+(1./X-1./(Y**2*X**3))*
     +SIN(2.*X*Y)+.5/(Y*X**2)
      RETURN
      END
      FUNCTION FUNCC(K,Y1,XL,X1,D,X,Y)
      CAC=XL*(X+X1)
      IF(CAC.GT.630) CAC=630
      CAP=XL*X1*2.0
      IF(CAP.GT.630) CAP=630
      EZ1=EXP(-CAC)
      EZ2=EXP(-CAP)
      CAD=XL*(X1-X)
      IF(CAD.GT.630) CAD=630
      FUNCC=2./Y1*D*SIN(XL*Y)*(EXP(-CAD)-EZ1)
     +/(1.-EZ2)
      RETURN
      END
      FUNCTION FUNDD(K,Y1,XL,X1,D,X,Y)
      CA=XL*X
      IF(CA.GT.450) CA=450
      FUNDD=-2./(Y1*TANH(XL*X1))*D*(SINH(CA)-TANH(XL*X1)*
     +COSH(CA))*SIN(XL*Y)
      RETURN
      END
      FUNCTION FUNCEX(K,Y1,XL,X1,D,X,Y)
      CAC=XL*(X+X1)
      IF(CAC.GT.630) CAC=630
      CAP=XL*X1*2.0
      IF(CAP.GT.630) CAP=630
      CAD=XL*(X1-X)
      IF(CAD.GT.630)  CAD=630
      EZ1=EXP(-CAC)
      EZ2=EXP(-CAP)
      FUNCEX=2./Y1*D*SIN(XL*Y)*(EXP(-CAD)+EZ1)*XL
     +/(1.-EZ2)
      RETURN
      END
      FUNCTION FUNCEY(K,Y1,XL,X1,D,X,Y)
      CAC=XL*(X+X1)
      IF(CAC.GT.630) CAC=630
      CAP=XL*X1*2.0
      IF(CAP.GT.630) CAP=630
      CAD=XL*(X1-X)
      IF(CAD.GT.630) CAD=630
      EZ1=EXP(-CAC)
      EZ2=EXP(-CAP)
      FUNCEY=2./Y1*D*COS(XL*Y)*XL*(EXP(-CAD)-EZ1)/
     +(1.-EZ2)
      RETURN
      END
      FUNCTION FUNDEX(K,Y1,XL,X1,D,X,Y)
      CA=XL*X
      IF(CA.GT.450) CA=450
      FUNDEX=-2./(Y1*TANH(XL*X1))*D*(COSH(CA)-TANH(XL*X1)
     +*SINH(CA))*XL*SIN(XL*Y)
      RETURN
      END
      FUNCTION FUNDEY(K,Y1,XL,X1,D,X,Y)
```

```
      CA=XL*X
      IF(CA.GT.450) CA=450
      FUNDEY=-2./(Y1*TANH(XL*X1))*D*(SINH(CA)-TANH(XL*X1)*
     +COSH(CA))*COS(XL*Y)*XL
      RETURN
      END
```

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PART III

Finite Analytic Numerical Solution of Two-Dimensional Navier-Stokes Equations in Primitive Variables

ABSTRACT

A numerical scheme called the 'Finite Analytic Method' is used to solve the two dimensional Navier-Stokes equations. The basic idea of this method, which was developed in the last three years, is to obtain local analytic solutions and use them in the numerical solution of any partial differential equation, linear or non-linear. The flow region is subdivided into a number of small rectangular subregions, in which the Navier-Stokes equations are linearized and an analytical solution obtained. When the local analytic solution is evaluated at an interior point of an element a linear algebraic equation is obtained relating the interior nodal value with the neighboring nodal values. The local finite analytic solutions for all elements are overlapped to cover the entire flow region. While the behavior of the non-linearity of the Navier-Stokes equations is preserved, a set of linear algebraic equations result from the analytic solutions. This set of linear algebraic equations is then solved iteratively to provide the numerical solution to the total problem.

A general 9-point Finite Analytic (FA) formula is developed for the Navier-Stokes equation in a finite element. The Navier-Stokes equations are formulated using the primitive variables. A new iterative scheme which solves the continuity equation, Poisson pressure equation and the momentum equations (i.e., x- and y-momentum equations) for the three primitive variables is devised. The FA numerical solution

is first obtained for stagnation point flow and a comparison with the exact solution is made. Then the formula is used to obtain the numerical solution for a flat plate-wake combined problem and also for a square driven cavity flow. The results are obtained for Reynolds numbers 100, 400, and 800.

It is shown from the above example that the FA numerical solution converges rapidly and the FA method gives accurate and stable numerical solution.

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NOMENCLATURE

A,B	Linearized convective coefficients in the momentum equations
a,b,c	Coefficients of polynomials representing boundary conditions
$a_0,a_1,\ldots,a_8$	Coefficients of second degree polynomial in x and in y
$A_n(x)$	Function of x
$B_m(y)$	Function of y
C_n	Constants; also coefficients of 9-point FA formulae
C_j	Fourier constants
D_n	Constants
$D(y)$	Function of y
D	Dilation
E	Element
$E_n,\bar{E}_n$	Integrals; $n = 1,2,3$
$F_n,\bar{F}_n$	Integrals; $n = 1,2,3$
f_n	Function representing boundary condition
$f(x,y)$	Function of x and y representing the nonhomogeneous term in Poisson equation
G	Nonhomogeneous term of an elliptic partial differential equation
G_n	Integrals; $n = 1,2,3$
h	Grid size in x-direction ($= \Delta x$)
I	Integral
i,j	Nodes in the x,y directions

IMAX,JMAX	Maximum values of nodes i,j
k	Grid size in y-direction (= Δy)
$L(\phi)$	Operator of ϕ
L	Reference length scale
l,m,n	Dummy variables in series summation
P	Pressure
p	Dimensionless pressure
q_n, q_m	Function of A,B and eigenvalue
R	Flow region
Re	Reynolds number
U	Velocity in x-direction
U_r	Reference velocity scale
u	Nondimensional velocity in x-direction
$\bar{u}$	Velocity in transformed coordinates
u_{aV}	Average velocity in x-direction
V	Velocity in y-direction
v	Nondimensional velocity in y-direction
v_{aV}	Average v velocity in y-direction
X,Y	Cartesian coordinate system
x,y	Dimensionless Cartesian coordinate system
X(x)	Function of x
Y(x)	Function of y

Greek Symbols

ρ	Density of fluid
μ	Viscosity of fluid
μ_n	Eigen values

ν	Kinematic viscosity of fluid
ν_m	Eigen values
$\lambda_l, \lambda_m, \lambda_n$	Eigen Values
∇^2	Laplacian
ψ	Stream function

Subscripts

a,b	Parts of total solution
E,S,N,W	East, south, north and west boundaries in an element
i,j	Nodes in the x,y directions
NE	North-east (similarly for NW, SE, SW, NC, SC, EC, WC)
P	Interior node
x,y	Derivatives in x,y
0	Node on the wall
1,2	Parts of total solution
1,2	First, second nodes closest to the boundary of the total region

CHAPTER 1

INTRODUCTION

The Navier-Stokes equations are a unique set of equations in the sense that only a handful of exact solutions of these equations are available. This is mainly due to the non-linearity of the equations and the coupling of variables with partial differential equations of higher order. In addition, it is often required to be solved for complex geometry and boundary conditions. Therefore, the numerical solutions of the Navier-Stokes equations governing the flow of a viscous imcompressible fluid have been the subject of many studies during the last few decades. In case of two dimensional flow, there are two ways of formulating these equations, namely, the vorticity-streamfunction formulation and the primitive variable (p,u,v) approach. For a two dimensional flow, there are two coupled governing equations, one linear and one non-linear (or quasi-linear) to be solved using the vorticity-streamfunction approach and three coupled equations, one linear and two non-linear (or quasi-linear),to be solved using the primitive variable method.

In the past, many investigators had solved Navier-Stokes equations numerically with the vorticity-streamfunction formulation for two dimensional incompressible laminar flows. The obvious advantage for this choice was that there are only two coupled equations for vorticity and streamfunction to be solved. The third variable, namely pressure, can be solved afterwards. This formulation, however, has a disadvantage in that it is not easily extendable to turbulent flows and three dimensional flow applications. Another disadvantage with this method is that there is difficulty in specifying the vorticity boundary condition. With this method of calculation, it is possible to obtain the velocities from the stream function. The pressure distribution is calculated once the velocity is known. The primitive variable approach, on the other hand, has more unknowns and equations to be solved simultaneously. The difficulties in solving primitive variable approach numerically are first the conservation of mass cannot be easily satisfied, and second the numerical solution is relatively unstable. Therefore, it requires various schemes to stabilize the numerical solution. However, it is preferred over the vorticity-streamfunction method as the pressure and velocity variables have more practical value than vorticity or streamfunction. What is more important, the primitive variable approach can be extended to three dimensional laminar or turbulent flow.

In the present work, the recently developed finite analytic method [1] is employed for solving Navier-Stokes equations formulated in primative variables. Before introducing this method, a brief review of other numerical schemes is done. One of the most widely used method is the finite difference scheme. In this scheme either a forward difference or a backward difference or a central difference formula is used to replace a derivative in the governing equation. Of the three different formulae, the central difference formula has a better accuracy and is preferred over the other two. However, it cannot be used near the boundary as an extra node has to be located outside the boundary of the flow. It is also found that the use of the central difference formula for convective term for high Reynolds number flow may develop numerical instability [2]. This difficulty is partly overcome by introducing the upward (or upwind) differencing method which shifts the difference scheme toward the upstream. The upwind difference scheme, however, produces large numerical diffusion and must be made judiciously at a given Reynolds number.

Another method widely used in the calculation of Navier-Stokes equations is the finite element method. This method considers an approximate function which is often a polynomial of low degree in a small element of the flow. However, the approximate functions in general cannot satisfy the governing equation exactly. The approximate functions are made to satisfy the governing equation in an integral sense by the weighted residue method or variational principle. The integral form results in algebraic equations which are then solved iteratively. This method seems to produce more stable results than the finite difference but it is not problem free. Problems of accuracy and stability still remain, particularly when the flow of high Reynolds number is considered.

In the present study, the method of numerical computation used is a method recently developed by Li and Chen [1]. This method is called the Finite Analytic (FA) method. In this method an analytic solution is obtained in each element of the flow region which is then evaluated at the interior node. This results in a set of algebraic equations which is then solved iteratively by any of the iterative schemes available. It was shown by Chen and Naseri and Li [2] that the Finite Analytic method is more accurate and stable than the other numerical methods.

The FA method will be used to solve the Navier-Stokes equations formulated in the primitive variables of u, v and p.

In Chapter 2, the basic principle of the FA method is described. In Chapter 3, the solution of primitive variables for the Navier-Stokes equations is obtained by the FA method. The flow chart and the method of computation for the numerical solutions are given in Chapter 4. Then in Chapter 5, a simple case of stagnation flow which has an exact solution is considered as an example to verify the FA solutions obtained in Chapter 3. In Chapters 6 and 7, this FA method is used to obtain solutions for flow over a flat plate and in a cavity. The detailed derivations of the solutions in Chapter 3 are given in Appendices A, B, and C. The computer program is given in Appendix D.

CHAPTER 2

PRINCIPLE OF THE FINITE ANALYTIC METHOD

In the Finite Analytic (FA) method of solution, the total flow region R under consideration (fig. 2.1) is divided into a number of small rectangular or square subregions called elements. In each of these elements, the partial differential equation (PDE) governing the flow is solved analytically. If the PDE is non-linear, it is linearized in each of the small elements and analytical solutions are obtained in those small elements. The local analytic solution is then evaluated at an interior node and the FA solution is written in the form of an algebraic equation relating the evaluated, interior nodal value to its neighboring nodal points. By grouping these FA solutions of all the elements which overlap to cover the entire flow region as shown in dashed line in fig. 2.2, a system of linear algebraic equations is obtained. These equations are then solved iteratively to provide the numerical solution in the total flow region R.

As an example, a general elliptic PDE $L(\phi) = G$ is considered where L is any linear or non-linear operator. When the boundary conditions are properly specified, the problem is well posed. If the entire problem had an analytic solution, a numerical method of solution would have been unnecessary. However, for most engineering problems, due to the non-linearity of the equations or the complexity of the geometries and boundary conditions, analytic solutions cannot be

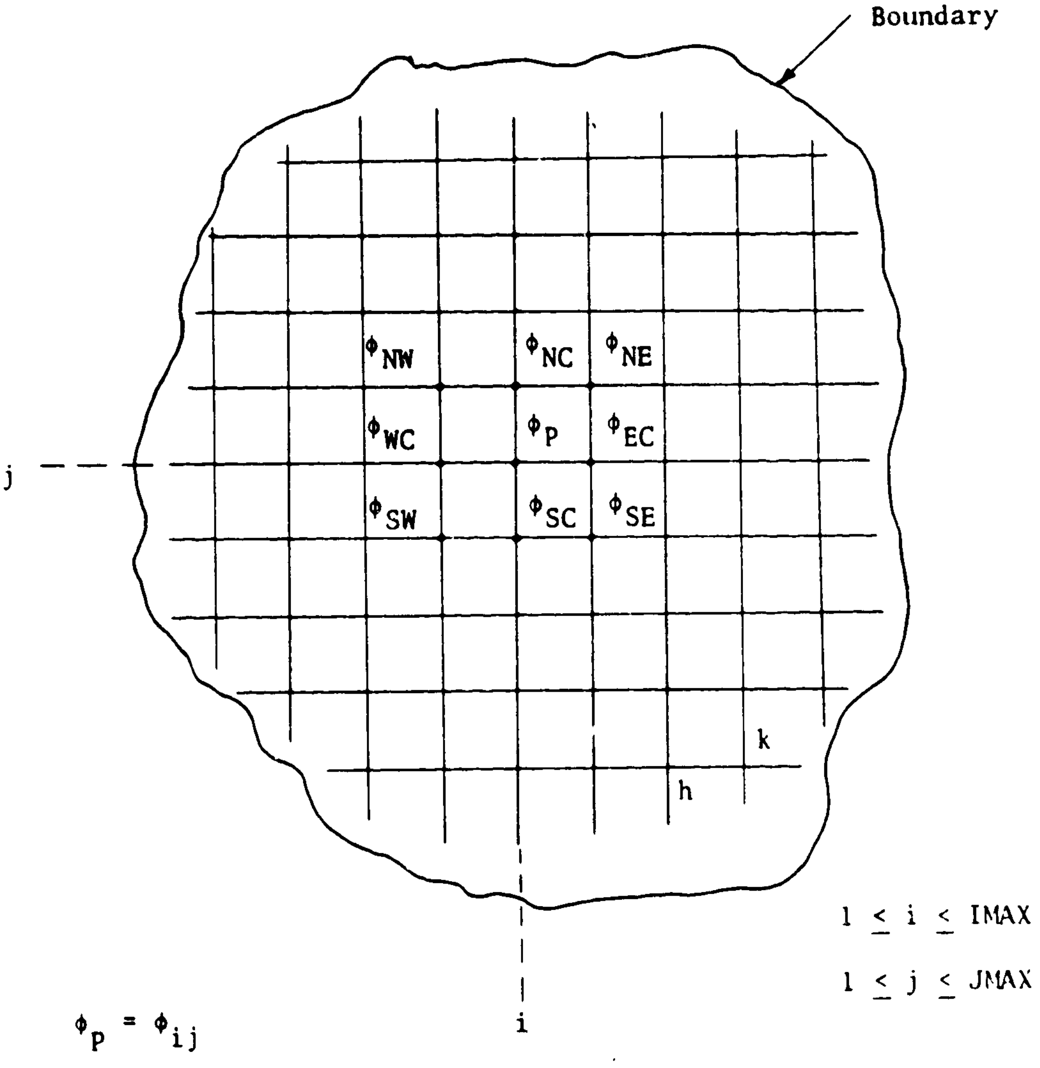

Figure 2.1 Region R

obtained. Therefore, numerical techniques are used to solve these problems.

In the FA method, the geometry whether complex or simple is broken into a number of small elements and the PDE is solved analytically in these small elements. As shown in fig. 2.1, the region R is subdivided into smaller regions or elements by passing horizontal and vertical lines through the region. These lines intersect at points (i,j) where i = 1,2,3,...,IMAX and j = 1,2,3,...,JMAX. To find the solution at any node (i,j), a region enclosed by the eight nodes (i+1,j+1), (i+1,j), (i+1,j-1), (i,j-1), (i-1,j-1), (i-1,j), (i-1,j+1) and (i,j+1) is considered. These notations for the nodes are abbreviated as NE (north-east), EC (east-central), SE (south-east), SC (south-central), SW (south-west), WC (west-central), NW (north-west) and NC (north-central), respectively.

The problem is now reduced to one having many finite elements where analytic solutions are sought. However, even after breaking up the complex geometry of region R into small elements, the analytic solution may still be difficult to obtain as is the case with non-linear PDE like the Navier-Stokes equations. In this situation, the non-linear terms of the equation are locally linearized in each of the elements. For example, the non-linear convective terms in the N-S equation can be locally linearized by taking the convective velocity components as an averaged velocity of the local elements. Since the local linearization is applied only to a small finite region, the overall non-linear effect is still preserved by changing the convective velocity in each element.

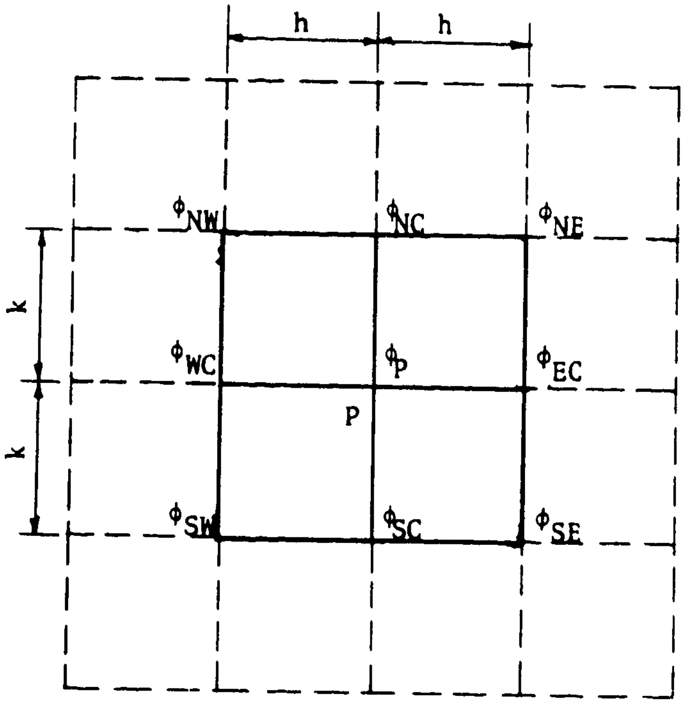

Figure 2.2 A Typical Element

Now, consider a local element E as shown in fig. 2.2 and let the linearized PDE to be solved in this element be $L(\phi) = G$. The analytic solution when obtained is a function of the boundary conditions of this element. Therefore,

$$\phi = \phi[f_N(x), f_S(x), f_E(y), f_W(y), x, y, h, k, G], \tag{2.1}$$

where $f_N(x)$, $f_S(x)$, $f_E(y)$ and $f_W(y)$ are the boundary conditions on the northern, southern, eastern and western sides of the element, x and y are the independent variables, h and k are the grid sizes in the x and y directions and G is the non-homogeneous term. The functions representing the boundary conditions can be approximated by polynomials of second degree or other suitable functions. For example, the northern boundary condition can be written as

$$f_N(x) = a_N + b_N x + C_N x^2 , \tag{2.2}$$

where the coefficients a_N, b_N and c_N can be expressed in terms of the three nodal values of ϕ, namely ϕ_{NW}, ϕ_{NC} and ϕ_{NE}. Thus

$$f_N(x) = f_N (\phi_{NW}, \phi_{NC}, \phi_{NE}, x). \tag{2.3}$$

Similarly for the other sides, the boundary conditions are

$$f_S(x) = f_S (\phi_{SW}, \phi_{SC}, \phi_{SE}, x), \tag{2.4}$$

$$f_E(y) = f_E(\phi_{NE}, \phi_{EC}, \phi_{SE}, y), \tag{2.5}$$

$$f_W(y) = f_W(\phi_{NW}, \phi_{WC}, \phi_{SW}, y). \tag{2.6}$$

With these boundary conditions, an analytic solution is obtained for the element under consideration. To evaluate ϕ at the interior node P, the values of x and y are substituted in equation (2.1). This gives

$$\phi_p = \phi_p(\phi_{NE}, \phi_{NC}, \phi_{NW}, \phi_{EC}, \phi_{WC}, \phi_{SE}, \phi_{SC}, \phi_{SW}, G). \tag{2.7}$$

This is the fundamental formula for the FA method. From this, an algebraic expression is obtained as

$$\phi_p = C_{NE}\phi_{NE} + C_{NC}\phi_{NC} + C_{NW}\phi_{NW} + C_{EC}\phi_{EC} + C_{WC}\phi_{WC} + C_{SE}\phi_{SE} + C_{SC}\phi_{SC} + C_{SW}\phi_{SW} + F(G). \tag{2.8}$$

Equation (2.8) is the 9-point FA solution to the PDE.

At this point, it is worth mentioning that equation (2.8) gives the exact solution for the point p in the element in the sense that it is obtained from an analytic solution to the linearized PDE in the finite element E. On the other hand, in the finite difference method, each derivative in the PDE is approximated using Taylor's series expansion of the dependent variable about its neighboring points thereby committing the truncation error. This significantly reduces the accuracy of the solution obtained from the finite difference method.

Another important feature of the FA solution is that, if it is required to find the derivative of ϕ at the node p, i.e., $(\partial\phi/\partial x)|_p$, the only thing to be done in the FA solution is to differentiate equation (2.1) with respect to x and substitute the values of x and y in the resulting expression without loss of accuracy . A truncation error is introduced further if the derivatives are obtained by the finite difference method.

In the internal small elements of the total flow region R, the surrounding eight nodal points such as ϕ_{NE}, ϕ_{EC}, etc. in equation (2.8) are unknowns. However, each is, in turn, expressed as an analytic function of its surrounding nodal points. Equation (2.8) is thus used to express all the unknown nodes in the whole region R. The system of linear algebraic equations is then formed which is solved numerically using any of the iterative methods available. It should be remarked here that first the FA solutions for two adjacent nodal values are obtained from two elements which are overlapping each other. Secondly, the algebraic equations are obtained from the well posed analytic solution. Therefore the FA solution is expected to be numerically stable.

This is the basic principle of the FA method of solution which will be used subsequently in solving the Navier-Stokes equations.

CHAPTER 3

FA SOLUTION OF 2D NAVIER-STOKES EQUATIONS

In this chapter, the FA solutions of the Navier-Stokes equations for u, v and p are obtained. The two Navier-Stokes equations for u and v along with the continuity equation are to be solved for u, v and p. Though the number of equations and the number of unknowns are equal, the pressure variable is difficult to solve in the conventional form. Therefore, it is more convenient to solve the equations if the problem is formulated in a slightly different way. The following discussion gives the formulation of the problem and then the solution.

3.1 Formulation of the Problem

For a two-dimensional, steady, incompressible flow, the Navier-Stokes equations are

$$\rho(UU_x + VU_y) = -P_x + \mu(U_{xx} + U_{yy}) \ , \tag{3.1}$$

$$\rho(UV_x + VV_y) = -P_y + \mu(V_{xx} + V_{yy}) \ , \tag{3.2}$$

and the continuity equation is

$$U_x + V_y = 0 \ . \tag{3.3}$$

Here the fluid has a density ρ and a constant coefficient of viscosity μ. The above equations can be made non-dimensional with the following variables

$$u = \frac{U}{U_r};\ v = \frac{V}{U_r};\ p = \frac{P}{\rho U_r^2};\ x = \frac{x}{L};\ y = \frac{y}{L}\ ,$$

where U_r and L are some reference velocity and length scales, respectively. These quantities are substituted into the above equations and the resulting equations are

$$uu_x + vu_y = -p_x + \frac{1}{Re}(u_{xx} + u_{yy}), \tag{3.4}$$

$$uv_x + uv_y = -p_y + \frac{1}{Re}(v_{xx} + v_{yy})\ , \tag{3.5}$$

$$u_x + v_y = 0\ , \tag{3.6}$$

where $Re = \frac{\rho U_r L}{\mu}$ is the Reynolds number. Equations (3.4), (3.5) and (3.6) have to be solved for the three unknowns, namely, u, v and p.

In order to solve the pressure variable, it is more convenient to cast the Navier-Stokes equations in the form of Poisson equation for pressure. This is accomplished by differentiating equation (3.4) with respect to x and equation (3.5) with respect to y and adding the two. The resulting equation is

$$p_{xx} + p_{yy} = (2u_x v_y + u_x^2 + v_y^2) + \frac{1}{Re}(D_{xx} + D_{yy})$$
$$- (uD_x + vD_y)\ , \tag{3.7}$$

where

$$D = u_x + v_y .$$

From equation (3.3), D = 0. Thus equation (3.7) reduces to

$$p_{xx} + p_{yy} = 2(u_x v_y - v_x u_y). \tag{3.8}$$

Now there are four equations to be solved, namely

$$p_{xx} + p_{yy} = 2(u_x v_y - v_x u_y), \tag{3.8}$$

$$uu_x + vu_y = - p_x + \frac{1}{Re} (u_{xx} + u_{yy}) , \tag{3.4}$$

$$uv_x + uu_y = - p_y + \frac{1}{Re} (u_{xx} + u_{yy}), \tag{3.5}$$

and

$$u_x + u_y = 0 . \tag{3.6}$$

There are, however, only three unknowns. To make the problem well posed, three independent equations must be chosen. The choice of the equations depends on the flow. This is discussed in detail in chapter 4 along with the method of computation. In addition, the problem is still not well posed without an adequate knowledge of the overall boundary conditions for u, v and p. The usual boundary condition for u and v is the no-slip and impermeable condition on a solid wall or known flow profiles at inlet and outlet of a given region. The pressure boundary condition, on the other hand, is more difficult but can be obtained through the use of momentum equation and Taylor series expansion for pressure. This is done in detail in Chapter 4.

In order to derive finite analytic solution for numerical computation, the local analytic solution is sought in each local element. It is thus necessary to specify the boundary conditions for each element. The boundary conditions for the three variables u, v and p in this investigation are expressed in terms of the eight boundary nodal values surrounding the element. These nodal values are, in general, unknown and interrelated. So the FA solutions of all the elements in the region R are coupled and eventually determined by the overall boundary conditions. Approximation of element boundary conditions is important for obtaining a proper solution. Improper or imprecise numerical treatment of boundary conditions invariably leads to unacceptable or unreliable solutions.

In the following sections, the FA solution to each of the equations in an element is derived.

3.2 Local FA Solution of Poisson Equation

The Poisson equation for pressure derived in section 3.1 is a linear, second order, nonhomogeneous partial differential equation. This equation is to be solved in each element of the total region R in fig. (3.1). A typical element E with the boundary conditions is shown in fig. (3.2).

Now, the problem is to solve the two dimensional Poisson equation,

$$p_{xx} + p_{yy} = 2(u_x v_y - v_x u_y), \tag{3.9}$$

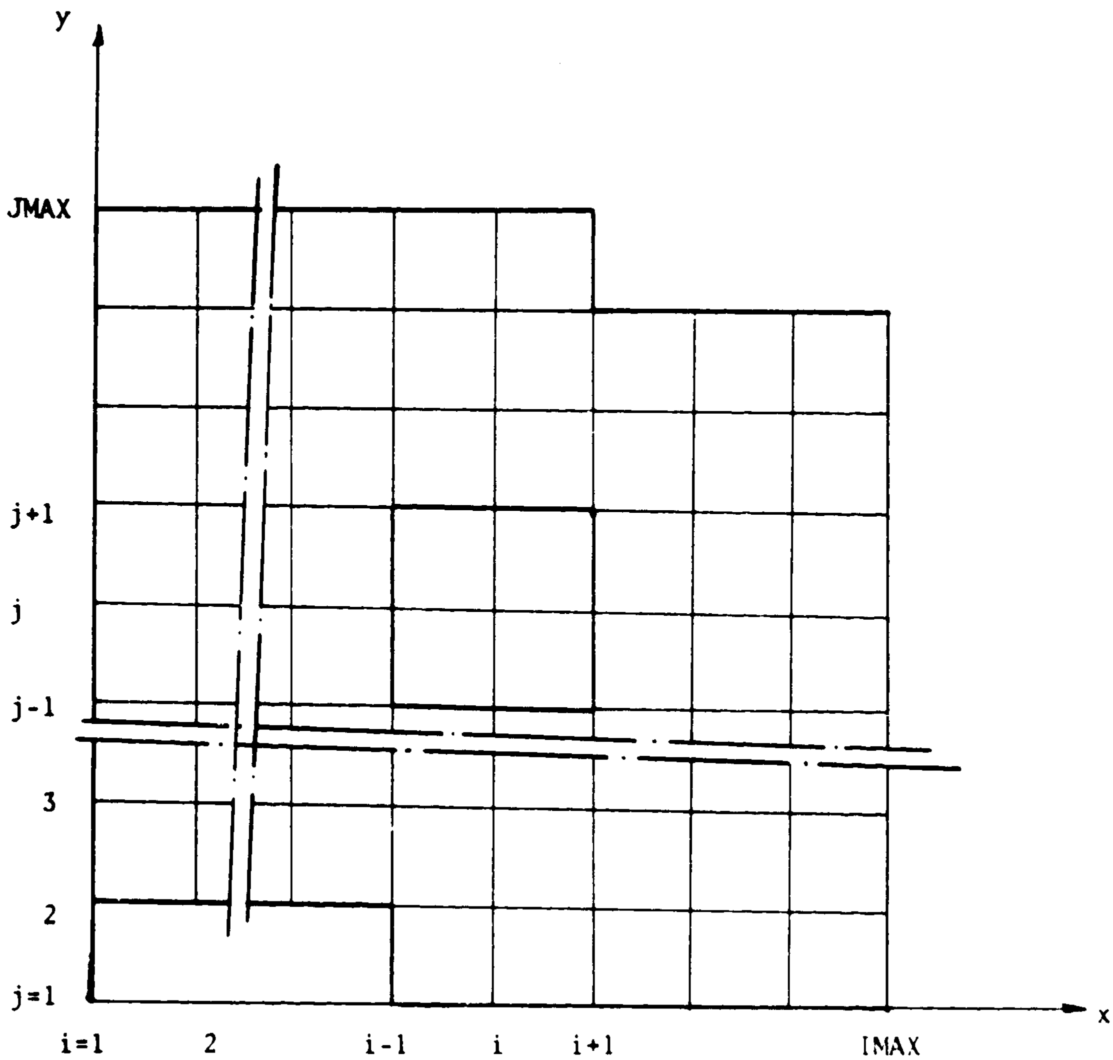

Figure 3.1 Flow Region R

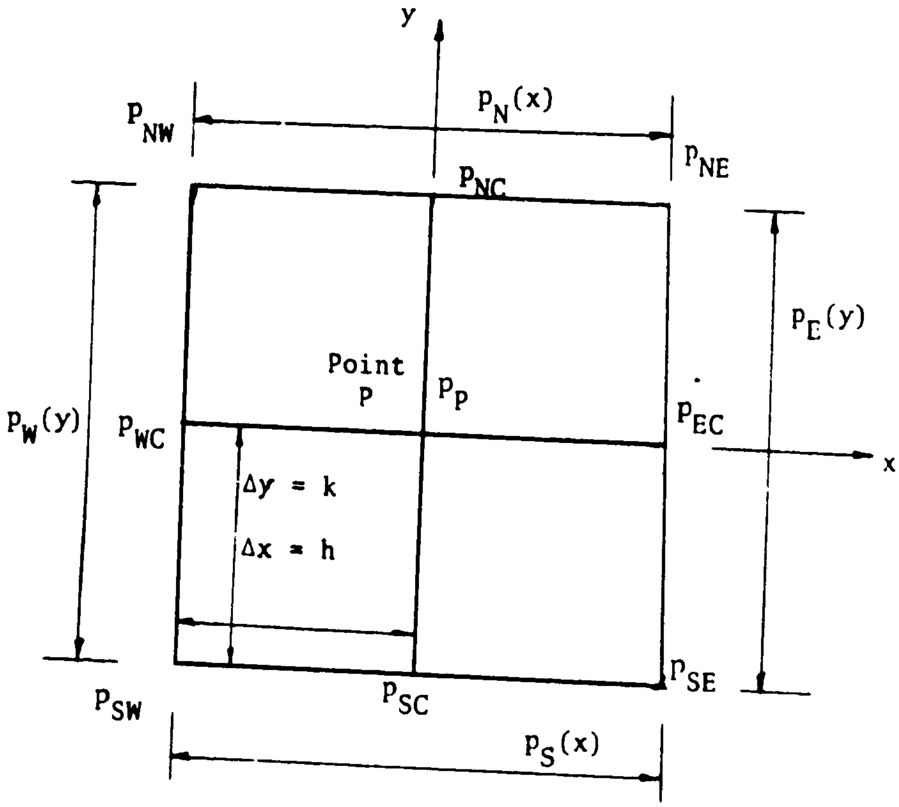

Figure 3.2 Typical Finite Analytic Element for Poisson Equation

in an element as shown in fig. (3.2) with the boundary conditions

$$\left.\begin{aligned} p(h,y) &= p_E(y) \\ p(-h,y) &= p_W(y) \\ p(x,k) &= p_N(x) \\ p(x,-k) &= p_S(x) \end{aligned}\right\} \quad . \tag{3.10}$$

In order to derive a 9-point finite analytic formula, the boundary conditions in the present study are approximately represented by second order polynomials in x or y. For example

$$p_E(y) = a_E + b_E y + c_E y^2 \ ,$$

where

$$\left.\begin{aligned} a_E &= p_{EC} \\ b_E &= \frac{1}{2k}(p_{NE} - p_{SE}) \\ c_E &= \frac{1}{2k^2}(p_{NE} - 2p_{EC} + p_{SE}) \end{aligned}\right\} \quad . \tag{3.11}$$

The other three boundary conditions are similarly written as

$$p_W(y) = a_W + b_W y + c_W y^2 ,$$

where

$$\left.\begin{aligned} a_W &= p_{WC} \\ b_W &= \frac{1}{2k}(p_{NW} - p_{SW}) \\ c_W &= \frac{1}{2k^2}(p_{NW} - 2p_{WC} + p_{SW}) \end{aligned}\right\} \quad . \tag{3.12}$$

and

$$p_N(x) = a_N + b_N x + c_N x^2 ,$$

where

$$\left.\begin{aligned} a_N &= p_{NC} \\ b_N &= \frac{1}{2h}(p_{NE} - p_{NW}) \\ c_N &= \frac{1}{2h^2}(p_{NE} - 2p_{NC} + p_{NW}) \end{aligned}\right\} ; \qquad (3.13)$$

and

$$p_S(x) = a_S + b_S x + c_S x^2 ,$$

where

$$\left.\begin{aligned} a_S &= p_{SC} \\ b_S &= \frac{1}{2h}(p_{SE} - p_{SW}) \\ c_S &= \frac{1}{2h^2}(P_{SE} - 2P_{SC} + P_{SW}) \end{aligned}\right\} . \qquad (3.14)$$

The nonhomogeneous term in the Poisson equation is assumed to be a function of x and y in the derivation. This function is then approximately expressed as a second degree polynomial in x and y and the coefficients of this polynomial are written in terms of the nodal values of the function.

Since the Poisson equation for pressure is linear, the problem is solved by dividing it into two simpler problems p_1 and p_2 and then super-imposing the results to obtain the final solution i.e., $p = p_1 + p_2$. The two simpler problems are

Problem (1): Homogeneous equation with nonhomogeneous boundary conditions, i.e.,

$$p_{1xx} + p_{1yy} = 0 \tag{3.15}$$

with

$$p_1 = p_E(y) \text{ at } x = h \quad ,$$
$$p_1 = p_W(y) \text{ at } x = -h \quad ,$$
$$p_1 = p_N(x) \text{ at } y = k \quad ,$$
$$p_1 = p_S(x) \text{ at } y = -k \quad .$$

Problem (2): Nonhomogeneous equation with homogeneous boundary conditions, i.e.,

$$p_{2xx} + p_{2yy} = 2(u_x v_y - v_x u_y) \tag{3.16}$$

with

$$p_2 = 0 \text{ at } x = \pm h \text{ and } y = \pm k.$$

Solution to Problem (1):

Again, for simplicity and due to linearity, the problem is divided into two parts, each having two homogeneous boundary conditions. Thus

$$p_1(x,y) = p_{1a}(x,y) + p_{1b}(x,y) \ , \tag{3.17}$$

where

$$p_{1axx} + p_{1ayy} = 0 \tag{3.18}$$

with the boundary conditions

$p_{1a} = p_E(y)$ at $x = h$,

$p_{1a} = p_W(y)$ at $x = -h$,

$p_{1a} = 0$ at $y = \pm k$;

and

$$p_{1bxx} + p_{1byy} = 0 \tag{3.19}$$

with the boundary conditions

$p_{1b} = 0$ at $x = \pm h$,

$p_{1b} = p_N(x)$ at $y = k$,

$p_{1b} = p_S(x)$ at $y = -k$.

The solutions for p_{1a} and p_{1b} are obtained by the method of separation of variables. These solutions are then superimposed to give the solution for p_1. The result is

$$p_1(x,y) = \sum_{n=1}^{\infty}[C_{1n}\text{Sinh } \mu_n x + C_{2n}\text{Cosh } \mu_n x] \text{ Sin } \mu_n(y + k)$$
$$+ \sum_{m=1}^{\infty}[C_{3m}\text{Sin } \nu_m y + C_{4m}\text{Cosh } \nu_m y] \text{ Sin } \nu_m (x + h), \tag{3.20}$$

where $\mu_n = n\pi/2k$ and $\nu_m = m\pi/2h$.

The constants in equation (3.20) are given in Appendix A along with the detailed derivation for p_1.

Solution to Problem (2)

As mentioned earlier, this problem is solved by expressing the nonhomogeneous term as a second degree polynomial in x and in y, i.e.,

$$f(x,y) = 2(u_x v_y - v_x u_y)$$

$$= a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^2 y^2$$

$$+ a_7 xy^2 + a_8 x^2 y \tag{3.21}$$

The nine coefficients in this polynomial are evaluated in terms of the nine nodal values of the function f(x,y). So the values of the coefficients are

$$a_0 = f_p$$
$$a_1 = \frac{1}{2h}(f_{EC} - f_{WC})$$
$$a_2 = \frac{1}{2k}(f_{NC} - f_{SC})$$
$$a_3 = \frac{1}{4hk}(f_{NE} - f_{NW} - f_{SE} + f_{SW})$$
$$a_4 = \frac{1}{2h^2}(f_{EC} - 2f_p + f_{WC})$$
$$a_5 = \frac{1}{2k^2}(f_{NC} - 2f_p + f_{SC})$$
$$a_6 = \frac{1}{4h^2k^2}(f_{NE} + f_{SE} + f_{NW} + f_{SW} - 2f_{EC} - 2f_{WC} - 2f_{NC} - 2f_{SC} + 4f_p)$$
$$a_7 = \frac{1}{4hk^2}(f_{NE} + f_{SE} - f_{NW} - f_{SW} - 2f_{EC} - 2f_{WC})$$
$$a_8 = \frac{1}{4h^2k}(f_{NE} + f_{NW} - f_{SE} - f_{SW} - 2f_{NC} + 2f_{SC}),$$

where the subscripts denote the value of f(x,y) at that node. With this polynomial approximation, the solution for p_2 with homogeneous boundary conditions is obtained as

$$p_2(x,y) = \sum_{n=1}^{\infty}[C_5 \text{Sinh}\, \lambda_n y + C_{6n} \text{Cosh}\, \lambda_n y + C_7 + C_8 y + C_9 y^2] *$$
$$\text{Sin}\, \lambda_n (x + h) \tag{3.22}$$

The constants in equation (3.22) are given in Appendix A along with the detailed derivation for p_2.

Equations (3.20) and (3.22) are added to give a solution for Poisson equation. So , with $\lambda_n = n\pi/2h$,

$$
\begin{aligned}
p(x,y) = \sum_{n=1}^{\infty} & [C_{1n}\text{Sinh}\,\mu_n x + C_{2n}\text{Cosh}\,\mu_n x]\,\text{Sin}\,\mu_n(y + k) \\
& + \sum_{m=1}^{\infty} [C_{3m}\text{Sinh}\,\nu_m y + C_{4m}\text{Cosh}\,\mu_m y]\,\text{Sin}\,\nu_m(x + h) \\
& + \sum_{n=1}^{\infty} [C_{5n}\text{Sinh}\,\lambda_n y + C_{6n}\text{Cosh}\,\lambda_n y + C_7 + C_8 y + C_9 y^2]\,\text{Sin}\,\lambda_n(x+h).
\end{aligned}
\tag{3.23}
$$

The FA 9-point formula for any point in the element is obtained by substituting the corresponding values of x and y in equation (3.23). To find the pressure at the center of the element, x = 0 and y = 0 are substituted in the above equation. This gives

$$
p_P = p(0,0) = \sum_{n=1}^{\infty} C_2\text{Sin}(\tfrac{n\pi}{2}) + \sum_{m=1}^{\infty} C_4\text{Sin}(m\pi/2) + \sum_{n=1}^{\infty} [C_6+C_7]\,\text{Sin}(n\pi/2).
\tag{3.24}
$$

This equation is written in terms of the nodal values of p(x,y) and f(x,y) by replacing the coefficients in the above equation by their expressions. Then the FA formula becomes

$$
\begin{aligned}
p_P = {} & C_{NE}p_{NE} + C_{EC}p_{EC} + C_{SE}p_{SE} + C_{NC}p_{NC} + C_{SC}p_{SC} \\
& + C_{NW}p_{NW} + C_{WC}p_{WC} + C_{SW}p_{SW} + C_{NE}'f_{NE} + C_{EC}'f_{EC} \\
& + C_{SE}'f_{SE} + C_{NC}'f_{NC} + C_P'f_P + C_{SC}'f_{SC} + C_{NW}'f_{NW} \\
& + C_{WC}'f_{WC} + C_{SW}'f_{SW} ,
\end{aligned}
\tag{3.25}
$$

where the finite analytic coefficients C_{NE}, C_{EC}, ..., C_{NE}', C_{EC}', in the above algebraic equation are given in Appendix A. For example,

$$C_{EC} = \sum_{n=1,3..}^{\infty} \left[\frac{16}{m^3\pi^3}\right] \frac{\mathrm{Sin}(m\pi/2)}{\mathrm{Cosh}(m\pi/2)} = 0.205315\ ,$$

$$C_{SE}' = 8h^2\left[\sum_{m=1,3...}^{\infty} \frac{1}{\mathrm{Cosh}(m\pi/2)}\left(\frac{1}{2} - \frac{32}{m^4\pi^4}\right) - \left(\frac{4}{m^2\pi^2} - \frac{32}{m^4\pi^4}\right)\frac{\mathrm{Sin}(m\pi/2)}{m^3\pi^3}\right]$$

$$= 0.001895\ h^2\ .$$

An important feature of this formulation that can be revealed with a careful examination of the nodal coefficients of the 'p' terms (C_{NE}, C_{EC},...) is that they are independent of the specific problem considered and hence are universal constants. So these coefficients can be used to solve any equation of the form $\nabla^2 p = f$. For the 9-point FA solution for the Poisson equation, this implies that they can be calculated once and for all and be used thereafter. Further, if the grid spacing h were assumed equal to k, a great simplification and reduction in computation may be achieved, since the constant coefficients involving the Sin $\mu_n k$ and Sin $\mu_m h$ terms become the same. The coefficients C_{NE}, etc. for the homogeneous part become universal constants while the coefficients C_{NE}, etc. for the nonhomogeneous terms are universal constants multiplied by h^2 or k^2 or hk. Thus, when h = k, the following schematic FA solution is obtained for the Poisson equation.

p_P =

0.044685	0.205315	0.044685
0.205315		0.205315
0.044685	0.205315	0.044685

x p_n +

0.001895 x h^2	0.01855 x h^2	0.001895 x h^2
0.01855 x h^2	0.21289 x h^2	0.01855 x h^2
0.001895 x h^2	0.01855 x h^2	0.001895 x h^2

x f_n

Here the numerical values in the block are the corresponding FA coefficients to be multiplied by their corresponding nodal values p_n or f_n. n denotes the nodal points (8 for p_n and 9 for f_n).

3.3 Local FA Solution of Momentum Equation

The momentum equation for u or v is a nonlinear (or quasi-linear) second order partial differential equation. Since an analytic solution for the whole region of flow is not available, the finite analytic method is one way of obtaining a numerical solution. As in the solution of Poisson equation, the flow region R in fig. (3.1) is divided into many elements with the boundary conditions specified in fig. (3.3). To simplify the solution, the grid spacings in the x- and y-directions are assumed to be uniform. Further, to solve the nonlinear momentum equation analytically in the element, the non-linear convective terms are locally linearized. This linearization is a reasonable approximation as long as the elements are quite small compared to the whole region.

In this section, the solution of the momentum equation for u is obtained. From this, the solution to the v-momentum equation is written by inspection since the two momentum equations are similar to each other. The u-momentum equation is written here for convenience.

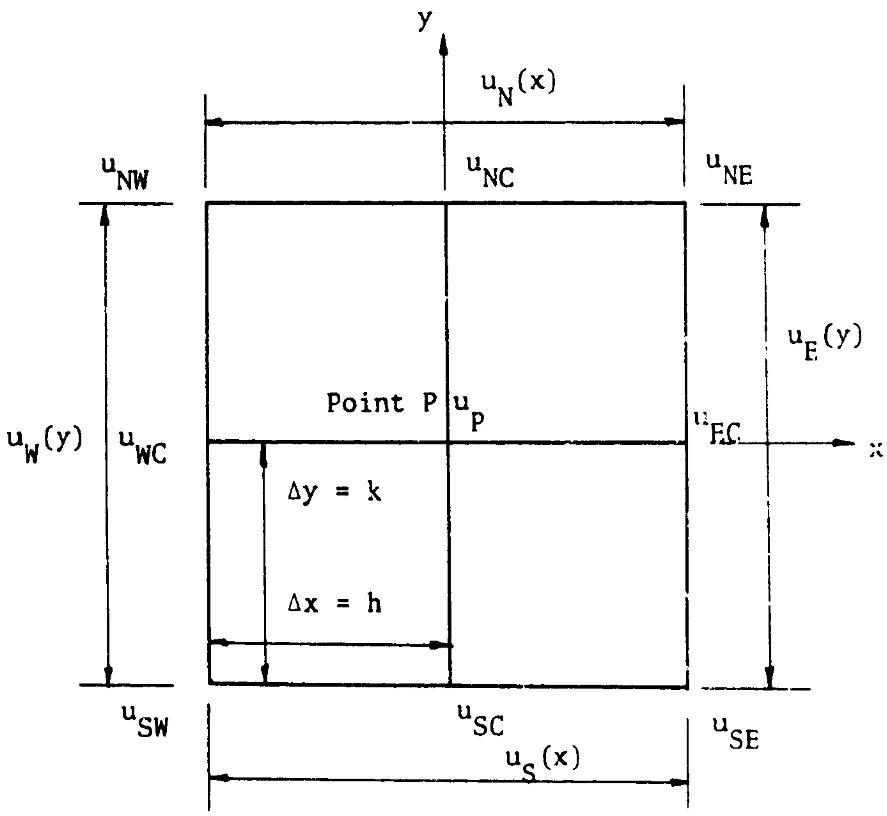

Figure 3.3 Typical Finite Analytic Element for Momentum Equations

$$(Reu)u_x + (Rev)u_y = -Rep_x + (u_{xx} + u_{yy}) \; , \tag{3.26}$$

$$Reu_{av} = 2A \text{ and } Rev_{av} = 2B \; , \tag{3.27}$$

where u_{av} and v_{av} are the averaged values of u and v in the element E. Substituting equation (3.27) for Reu and Rev, equation (3.26) is linearized to

$$2Au_x + 2Bu_y = u_{xx} + u_{yy} - Rep_x . \tag{3.28}$$

Equation (3.28) is now a PDE with constant coefficients. The boundary conditions for this equation are

$$\left.\begin{aligned} u(h,y) &= u_E(y) \\ u(-h,y) &= u_W(y) \\ u(x,k) &= u_N(x) \\ u(x,-k) &= u_S(x) \end{aligned}\right\} \; , \tag{3.29}$$

where $u_E(y)$, $u_W(y)$, $u_N(x)$ and $u_S(x)$ are each expressed by an appropriate function specified by three boundary nodal values (equations 3.33 - 3.36).

Introducing a change of variable

$$u = \bar{u}e^{(Ax + By)} \tag{3.30}$$

in equation (3.28), a simpler form of the momentum equation is obtained, i.e.,

$$(A^2 + B^2)\bar{u} = \bar{u}_{xx} + \bar{u}_{yy} - Re \; p_x e^{-(Ax + By)} . \tag{3.31}$$

The boundary conditions are

$$
\left.\begin{aligned}
\bar{u}(h,y) &= u_E(y)e^{-(Ah + By)} , \\
\bar{u}(-h,y) &= u_W(y)e^{(Ah - By)} , \\
\bar{u}(x,k) &= u_N(x)e^{-(Ax + Bk)} , \\
\bar{u}(x,-k) &= u_S(x)e^{-(Ax - Bk)} .
\end{aligned}\right\} \tag{3.32}
$$

The problem is to solve equation (3.31) with the boundary conditions (3.32) for an element. Since the momentum equation has been linearized, the above problem is split into simpler ones. The final solution is then obtained by superimposing the solutions of the simpler problems.

Before solving the problem, the boundary conditions (3.32) are expressed as second degree polynomials in x or y. The coefficients of these polynomials are written in terms of the surrounding nodal velocities. The eastern boundary condition is

$$u_E(y) = a_E + b_E y + c_E y^2 ,$$

where

$$
\left.\begin{aligned}
a_E &= u_{EC} \\
b_E &= \frac{1}{2k} (u_{NE} - u_{SE}) \\
c_E &= \frac{1}{2k^2} (u_{NE} - 2u_{EC} + u_{SE})
\end{aligned}\right\} . \tag{3.33}
$$

For the western boundary

$$u_W(y) = a_W + b_W y + c_W y^2 ,$$

where

$$a_W = u_{WC} \tag{3.34}$$

$$b_W = \frac{1}{2k}(u_{NW} - u_{SW})$$
$$c_W = \frac{1}{2k^2}(u_{NW} - 2u_{WC} + u_{SW}) \quad .$$

For the northern boundary

$$u_N(x) = a_N + b_N x + c_N x^2 ,$$

where

$$\left.\begin{aligned} a_N &= u_{NC} \\ b_N &= \frac{1}{2h}(u_{NE} - u_{NW}) \\ c_N &= \frac{1}{2h^2}(u_{NE} - 2u_{NC} + u_{NW}) \end{aligned}\right\} \; ; \qquad (3.35)$$

and for the southern boundary

$$u_S(x) = a_S + b_S x + c_S x^2 ,$$

where

$$\left.\begin{aligned} a_S &= u_{SC} \\ b_S &= \frac{1}{2h}(u_{SE} - u_{SW}) \\ c_S &= \frac{1}{2h^2}(u_{SE} - 2u_{SC} + u_{SW}) \end{aligned}\right\} \; . \qquad (3.36)$$

With these boundary conditions, the problem is split into three simpler problems. They are

Problem (1): Homogeneous equation with two homogeneous boundary conditions, i.e.,

$$(A^2 + B^2)\bar{u}_1 = \bar{u}_{1xx} + \bar{u}_{1yy} \tag{3.37}$$

with

$$\bar{u}_1(h,y) = (a_E + b_E y + c_E y^2)e^{-(Ah + By)} ,$$

$$\bar{u}_1(-h,y) = (a_W + b_W y + c_W y^2)e^{(Ah - By)} ,$$

$$\bar{u}_1(x,k) = 0 ,$$

$$\bar{u}_1(x,-k) = 0 .$$

Problem (2): Homogeneous equation with other two homogeneous boundary conditions, i.e.,

$$(A^2 + B^2)\bar{u}_2 = \bar{u}_{2xx} + \bar{u}_{2yy} \tag{3.38}$$

with

$$\bar{u}_2(h,y) = 0 ,$$

$$\bar{u}_2(-h,y) = 0 ,$$

$$\bar{u}_2(x,k) = (a_N + b_N x + c_N x^2)e^{-(Ax + Bk)} ,$$

$$\bar{u}_2(x,-k) = (a_S + b_S x + c_S x^2)e^{-(Ax - Bk)} .$$

Problem (3): Nonhomogeneous equation with homogeneous boundary conditions, i.e.

$$(A^2 + B^2)\bar{u}_3 = \bar{u}_{3xx} + \bar{u}_{3yy} - Rep_x e^{-(Ax + By)} \tag{3.39}$$

with

$$\bar{u}_3(h,y) = 0 ,$$

$$\bar{u}_3(-h,y) = 0 ,$$

$$\bar{u}_3(x,k) = 0 ,$$

$$\bar{u}_3(x,-k) = 0 .$$

The solutions to these three problems are finally superimposed to give the solution of the momentum equation i.e., $\bar{u} = \bar{u}_1 + \bar{u}_2 + \bar{u}_3$.

Solution to Problem (1)

Problem (1) is solved analytically using separation of variables. The solution is, with $\lambda_n = n\pi/2k$ and $q_n^2 = A^2+B^2+\lambda_n^2$,

$$\bar{u}_1(x,y) = \sum_{n=1}^{\infty}\{C_{1n}\mathrm{Sinh}(q_n x) + C_{2n}\mathrm{Cosh}(q_n x)\}\mathrm{Sin}\lambda_n(y+k)\,. \tag{3.40}$$

The constants in the above equation are given in Appendix B along with the detailed derivation for $\bar{u}_1(x,y)$.

Solution to Problem (2)

The solution to this problem is exactly similar to the solution to problem (1) (equation 3.40). If x,y,h,k,A,B and n in problem (1) are replaced by y,x,k,h,B,A and m, the solution of problem (2) is identical to that of problem (1). Therefore, the solution to problem (2) is

$$\bar{u}_2(x,y) = \sum_{m=1}^{\infty}\{C_{1m}\mathrm{Sinh}(q_m y) + C_{2m}\mathrm{Cosh}(q_m y)\}\ \mathrm{Sin}\ \mu_m(x+h) \tag{3.41}$$

The constants in equation (3.41) are given in Appendix B.

Solution to Problem (3)

The nonhomogeneous equation (3.39) can have different solutions depending on the way in which the nonhomogeneous term is expressed. Since this term represents the gradient of pressure in a small element, it may be assumed constant over the element without significantly affecting the accuracy of the solution. If, however, a very accurate result is required, the pressure gradient term can be expressed as a polynomial in x and y as was done for the nonhomogeneous term in Poisson equation. In the solution given below, this term is assumed constant. The reason is that the solution is much simpler and saves much computer time without any significant loss in accuracy.

The derivation is done in Appendix B. Here, only the solution is presented, which is, with $\lambda_\ell = \ell\pi/2h$,

$$\bar{u}_3(x,y) = \sum_{\ell=1}^{\infty} Y_\ell(y) \operatorname{Sin}\lambda_\ell (x+h) \quad , \tag{3.42}$$

where the function $Y_\ell(y)$ is

$$Y_\ell(y) = C_{3\ell} e^{q_\ell y} + C_{4\ell} e^{-q_\ell y} + C_{5\ell} e^{-By} \, . \tag{3.43}$$

The constants in equations (3.42) and (3.43) are given in Appendix B.

The three solutions equations (3.40), (3.41) and (3.42) obtained above are now combined to give the solution to the momentum equation. So

$$\bar{u}(x,y) = \bar{u}_1(x,y) + \bar{u}_2(x,y) + \bar{u}_3(x,y) \, . \tag{3.44}$$

But

$$u(x,y) = \bar{u}(x,y) e^{(Ax + By)} \, ,$$

thus

$$u(x,y) = [\bar{u}\ (x,y) + \bar{u}_2(x,y) + \bar{u}_3(x,y)] e^{Ax + By} \quad . \tag{3.45}$$

To calculate u at an interior node P, x = 0 and y = 0 are substituted in equation (3.45) to yield

$$u_P = \bar{u}_{1P} + \bar{u}_{2P} + \bar{u}_{3P} \quad . \tag{3.46}$$

$\bar{u}_{1p}$, $\bar{u}_{2p}$ and $\bar{u}_{3p}$ are evaluated from equations (3.40), (3.41) and (3.42) and substituted in (3.46). After some rearrangement, the expression for the velocity is obtained in the form

$$\begin{aligned} u_P = {} & C_{NE}U_{NE} + C_{EC}U_{EC} + C_{SE}U_{SE} + C_{NC}U_{NC} + C_{SC}U_{SC} \\ & + C_{NW}U_{NW} + C_{WC}U_{WC} + C_{SW}U_{SW} + C_p(Rep_x)_P \quad . \end{aligned} \tag{3.47}$$

This is the 9-point FA formula for the momentum equation where the subscript P refers to the quantity in parenthesis evaluated at the interior node P.

The finite analytic coefficients C_{NE}, C_{EC},..., in the above algebraic equation are given in Appendix A. Some of these coefficients are shown below

$$C_{NE} = \sum_{n=1,3}^{\infty} \left[\frac{1}{2\mathrm{Cosh}(q_n h)} \left\{\frac{e^{-Ah}}{2k^2}\left(E_2 + \frac{E_3}{k}\right) + \frac{e^{-Bk}}{2h^2}\left(\bar{E}_2 + \frac{\bar{E}_3}{h}\right)\right\} \mathrm{Sin}(n\pi/2)\right] ,$$

$$C_{EC} = \sum_{n=1,3}^{\infty} \left[\frac{1}{2\mathrm{Cosh}(q_n h)} \left\{\frac{e^{-Ah}}{k}\left(E_1 - \frac{E_3}{k^2}\right)\right\} \mathrm{Sin}(n\pi/2)\right] ,$$

$$C_p = \sum_{\ell=1}^{\infty} \frac{\lambda_\ell\left(e^{Ah} - e^{-Ah}(-1)^\ell\right)}{h(A^2 + \lambda_\ell^2)^2} \left[\frac{\mathrm{Sinh}(q_\ell - B)k + \mathrm{Sinh}(q_\ell + B)k}{\mathrm{Sinh}\,2q_\ell k} - 1\right] \mathrm{Sin}(\ell\pi/2) .$$

3.4 Solution of Continuity Equation

Like the Poisson equation and the momentum equation, the continuity equation can be solved analytically. The u-velocity given in equation (3.45) has been analytically calculated from the momentum equation. This analytical solution is substituted into the continuity equation

$$u_x + v_y = 0 \quad , \tag{3.6}$$

which gives

$$\begin{aligned} v_y = &- [\bar{u}_1(x,y) + \bar{u}_2(x,y) + \bar{u}_3(x,y)]_x e^{Ax + By} \\ &- [\bar{u}_1(x,y) + \bar{u}_2(x,y) + \bar{u}_3(x,y)] e^{Ax + By} \, . \end{aligned} \tag{3.48}$$

When equation (3.48) is integrated with respect to y, the solution for v velocity component in the FA element is obtained. If the integration is from the node SC ($y = -k$, $x = 0$) to the node p($y = 0$, $x = 0$), then the FA solution for v_p is connected to the eight neighboring u nodal values and v_{SC} nodal value. However, the integration can also be done from NC node ($y = k$, $x = 0$) to the node p($x = 0$, $y = 0$). Having integrated equation (3.48) and after some algebraic manipulation one has the solution for v_p as

$$v_p = 0.5\,(v_{SC} + v_{NC}) + C_{NE}C_{NE} + C_{EC}u_{EC} + \ldots + C_p u_p . \tag{3.49}$$

The coefficients C_{NE},... are quite different from those of equation (3.47). In fact, they are quite complicated and are, therefore, not used in the solution. These results are not presented here. Instead, the following approximate solution is used.

Since the continuity equation is much simpler than the momentum equation or the Poisson equation, it is solved using a polynomial approximation. The approximation involves assuming a polynomial for u,

$$u = a_0 + a_1x + a_2y + a_3xy + a_4x^2 + a_5y^2 + a_6x^2y^2 + a_7xy^2 + a_8x^2y \quad . \tag{3.50}$$

The coefficients in this equation are expressed in terms of the nodal values of u. Equation (3.50) is then differentiated with respect to x to give u_x. From resulting expression, u_x is calculated at all the nodes in the elements. With these values of u_x, a polynomial in x and y is written for u_x, i.e.

$$u_x = \bar{a}_o + \bar{a}_1x + \bar{a}_2y + \bar{a}_3xy + \bar{a}_4x^2 + \bar{a}_5y^2 + \bar{a}_6x^2y^2 + \bar{a}_7xy^2 + \bar{a}_8x^2y \, . \tag{3.51}$$

This equation is now substituted in the continuity equation and integrated with respect to y. Integration is performed once from the NC node and once from the SC node and the average of the two integration is obtained. The resulting solution for v at the node p is

$$v_p = 0.5\ (v_{NC} + v_{SC}) + 0.125\ (u_{NE} - u_{NW} - u_{SE} + u_{SW})\ . \tag{3.52}$$

The detail of the derivation of this equation is given in Appendix C.

CHAPTER 4

METHOD OF NUMERICAL COMPUTATION

In Chapter 3, the FA solutions for three different equations were obtained separately. In this chapter, these solutions will be arranged in a suitable way to obtain the complete solution of the problem in the total flow region, R. The method of numerical computation is shown in the flow chart in fig. 4.1.

4.1 Momentum Averaging Scheme

As mentioned in Chapter 3, there are four equations to be solved and only three unknowns. Obviously, these four equations are not all independent. Two out of the three equations (x- and y-momentum equations (3.4), (3.5) and the Poisson equation for pressure (3.8)) are independent. One way of making the problem well posed is to use only three equations at a time. The pressure is first calculated from the assumed velocity in the flow region using the Poisson equation. Then, in every element, the average of the assumed velocities for the element ($\bar{u}$ and $\bar{v}$) in the x- and y-directions are computed. If $\bar{u}$ is greater than $\bar{v}$, the x-momentum equation is used to obtain the velocity u in that element. Having done this, the continuity equation is used to obtain the other velocity component in the element, i.e., v. If, on the other hand, the average velocity in an element in the y-direction is greater, v is first calculated from the y-momentum equation and then u from the continuity equation.

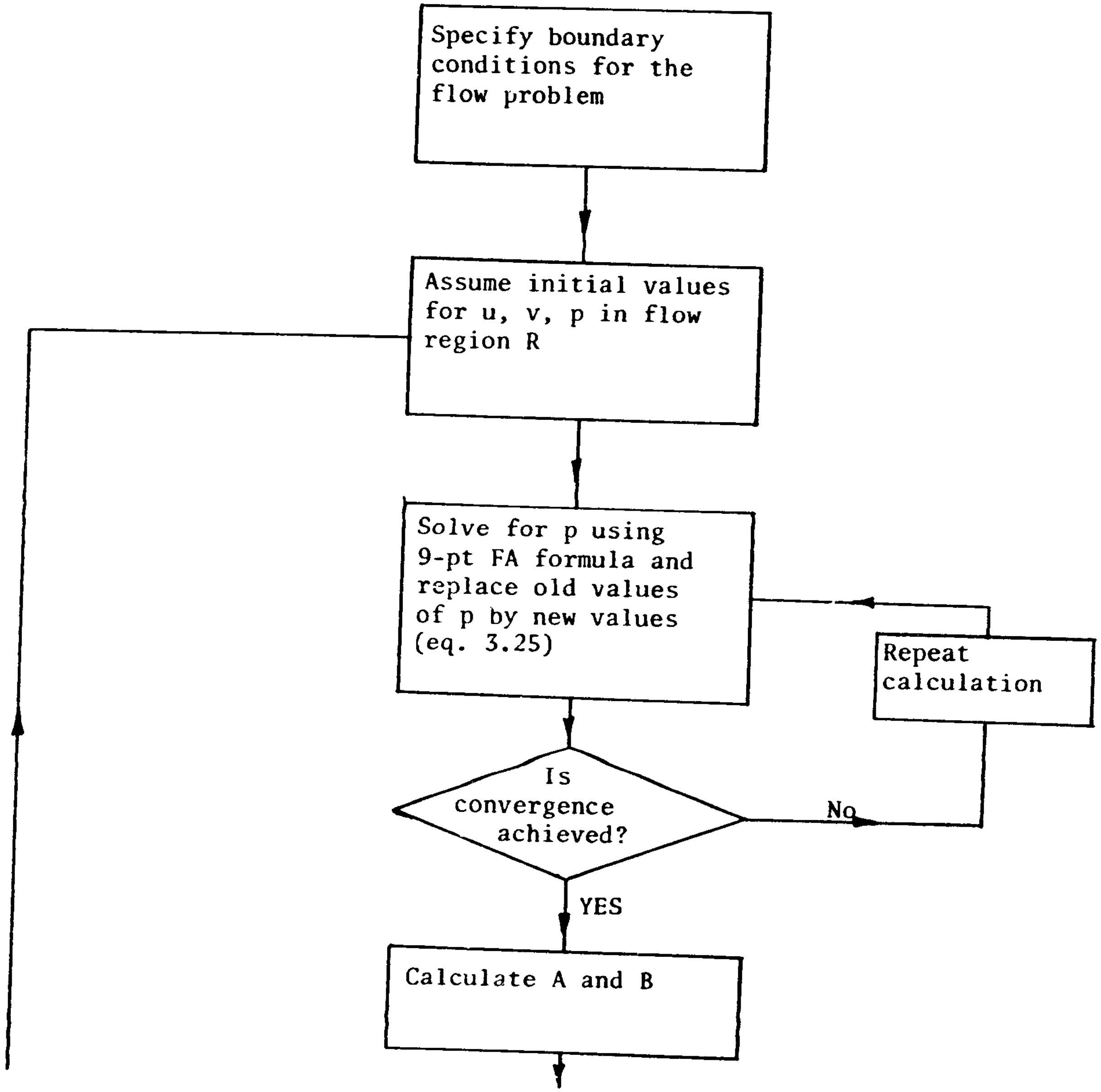

Figure 4.1 Flow Chart for Method of Numerical Computation.

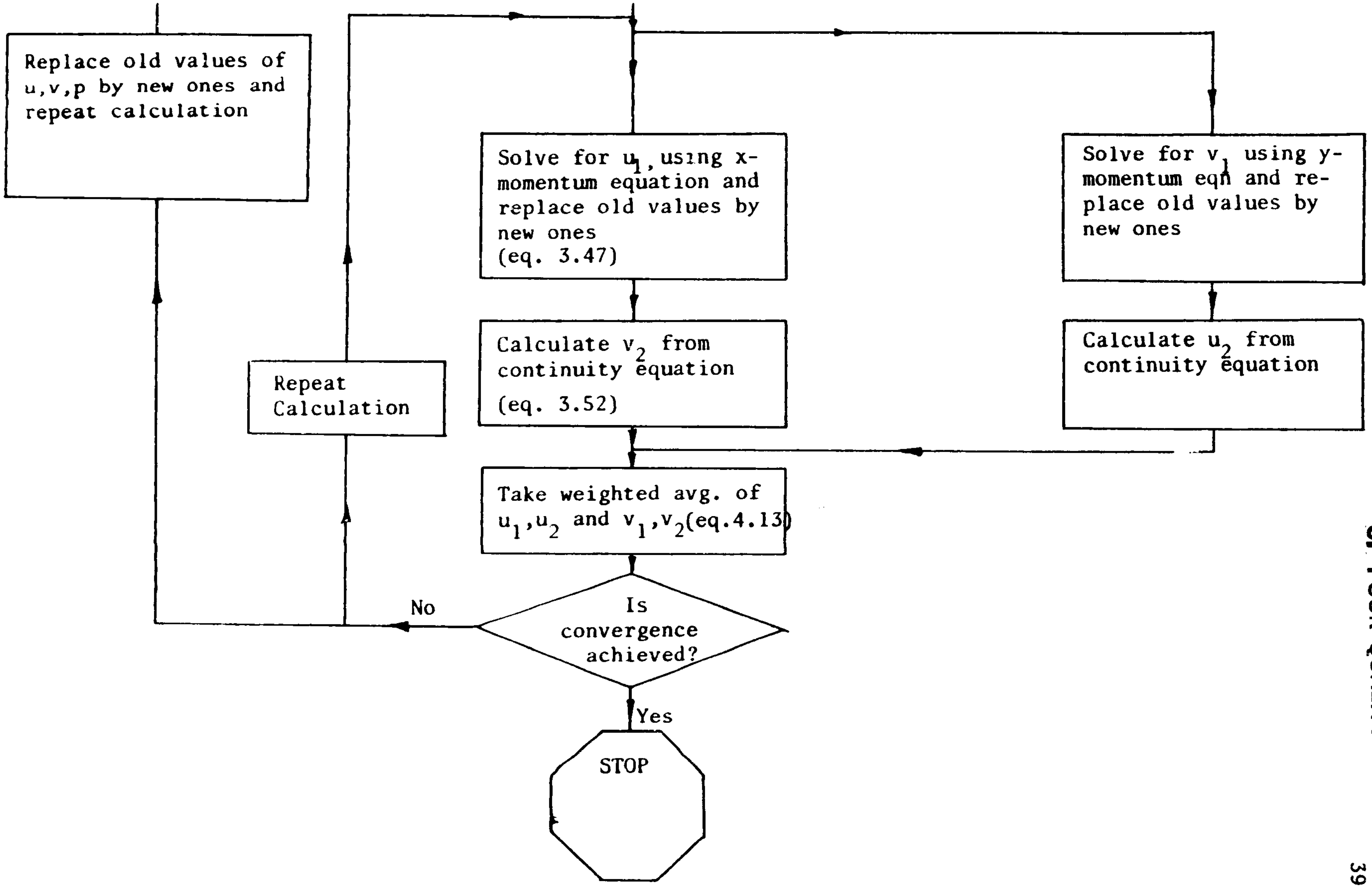

Figure 4.1 (Cont'd.)

C-4

This computational scheme called the Momentum Dominant Scheme seems quite reasonable to use but it has some disadvantages. It has a slow convergence. Furthermore, during iteration, if u is calculated from the x-momentum equation and v from the continuity equation, this value of v may not satisfy the y-momentum equation. Also, when u and v are of the same order, the two momentum equations are not used in the momentum dominant scheme.

In this study, the above scheme is slightly modified to give better convergence and more stable solution. The pressure is still calculated using equation (3.8). Next, the average velocities in each element are calculated from the previously obtained or assumed velocities. With these average velocities, the x- and y-momentum equations are both solved for u_1 and v_1 respectively. One set of velocities is obtained. Now the continuity equation is used to calculate the corresponding velocities v_2 with u_1 known and u_2 with v_1 known. Then a weighted average of the velocities u_1, u_2 and v_1, v_2 is calculated to give u and v in each element. With this new set of u and v, the Poisson equation is again solved for p and the whole process repeated till convergence is achieved.

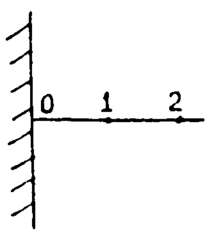

Figure 3.2 Pressure Boundary Condition

This computational scheme is the first of its kind in the solution of the Navier-Stokes equation and is called the momentum averaging scheme. In all earlier works, ref [3,4], the continuity equation is not independently used to calculate a variable in the solution scheme. The pressure is calculated from the Poisson equation and then the velocities u and v are calculated from the x- and y-momentum equations, respectively. The velocities calculated this way do not satisfy the conservation of mass criterion. To bring the effect of the continuity equation in the solution, the dilation term in the Poisson equation is not set to zero, though, theoretically speaking, this is zero for any incompressible flow field. It is stated in ref [3] that the retension of the temporal derivative of the local dilation in the Poisson equation for pressure is an essential requirement for the convergence of the numerical procedure. Any attempt to totally set to zero the dilation term leads to nonlinear instability in the numerical solution. This is not the case in the present method of solutions as the continuity equation is used independently as it should be to solve for u or v.

Since the solution technique is iterative in nature, some initial values for u,v and ρ need to be specified. For any Reynolds number, the initial values can all be set to zero. However, it is a better idea to use the results previously obtained, if any, for lower Reynolds number as the initial guess. For example, if the solution for Re = 1000 is required, then the solution for Re = 100 or Re = 400 can be used as the initial value in the solution for Re = 1000. This practice can save some computational time though the present FA method is stable with any initial value.

4.2 Pressure Boundary Condition

Depending on the geometry of the flow, boundary conditions for u, v and p are specified properly. Usually there is no difficulty in specifying the velocity boundary conditions. The pressure boundary condition, however, cannot be specified exactly. To specify pressure at the boundary, the first few terms of the Taylor series expansion for pressure are used depending on the accuracy required.

As an example, the wall in fig. 4.2 is considered. The pressures at interior points 1 and 2 are expanded in Taylor series as

$$p_1 = p_o + p_x\Big|_o \Delta x + p_{xx}\Big|_o \frac{(\Delta x)^2}{2!} + \dots \quad , \tag{4.1}$$

and

$$p_2 = p_o + p_x\Big|_o (2\Delta x) + p_{xx}\Big|_o \frac{(2\Delta x)^2}{2!} + \dots \quad . \tag{4.2}$$

Eliminating the second derivative from the above two equations gives

$$4p_1 - p_2 = 3p_o + p_x\Big|_o (2\Delta x) \quad , \tag{4.3}$$

or

$$p_o = \frac{1}{3}(4p_1 - p_2) - \frac{2}{3}\Delta x p_x\Big|_o \quad . \tag{4.4}$$

To determine $p_x\big|_0$, the x-momentum equation

$$uu_x + vu_y = -p_x + \frac{1}{Re}(u_{xx} + u_{yy}) \tag{4.5}$$

is evaluated at the point '0' in fig. (4.2). Since at the wall u = 0, v = 0 and $u_{yy} = 0$, equation (4.5) becomes

$$p_x\big|_0 = \frac{1}{Re}\, u_{xx}\big|_0 \; .$$

Hence,

$$p_0 = \frac{1}{3}(4p_1 - p_2) - \frac{2h}{3Re}\, u_{xx}\big|_0 \; . \tag{4.6}$$

u_{xx} is now obtained using Taylor series expansion for u. So

$$u_1 = u_0 + u_x\big|_0 (h) + u_{xx}\big|_0 (h)^2/2! + u_{xxx}\big|_0 \frac{h^3}{3!} \tag{4.7}$$

$$u_2 = u_0 + u_x\big|_0 (2h) + u_{xx}\big|_0 \frac{(2h)^2}{2!} + u_{xxx}\big|_0 \frac{(2h)^3}{3!} + \ldots \; . \tag{4.8}$$

Eliminating the third derivative in u gives

$$8u_1 - u_2 = 2h^2\, u_{xx}\big|_0 \; . \tag{4.9}$$

Since $u_0 = 0$ and $u_x\big|_0 = -v_y\big|_0 = 0$

$$u_{xx}\big|_0 = \frac{8u_1 - u_2}{2h^2} \; .$$

Hence,

$$P_0 = \frac{1}{3}(4p_1 - p_2) - \frac{(8u_1 - u_2)}{3Reh} \quad . \tag{4.10}$$

The boundary pressure on the other walls can be likewise derived.

4.3 Numerical Procedure

STEP 1. With the initial guess given and the boundary conditions specified, the 9-point FA formula for Poisson equation is first used to calculate the pressure in region R. The pressure at any node (i,j) is written in terms of the surrounding nodal values. This is done for all the nodes (2,j), (3,j),...,(i,j),...,(IMAX-1,j). Thus a system of algebraic equations is obtained which is then solved implicitly by the line by line implicit method using a tridiagonal solution scheme. This is repeated for all the lines starting from j=2 to j= JMAX-1. At each line, the 'TRIDAG' subroutine is called in the main program to solve for the unknown pressure implicitly. In this way, the solution for pressure in the whole flow region is obtained. Using these new values of pressure, the whole calculation process is repeated until the solution converges to desired accuracy. This iterative procedure within the equation is called an internal iteration. The number of internal iterations required for convergence is, in general, proportional to the number of nodes in a line.

STEP 2. Having calculated the pressure in the total flow region, the next step is to calculate the velocities at each of the elements. Before this is done, the average velocities in the x-direction and the y-direction are calculated from the initial guess or previous calculation.

There are various ways of doing this. For example, the average velocity for u is written as

$$u_{AV} = a_{NE}u_{NE} + a_{EC}u_{EC} + a_{SE}u_{SE} + a_{NC}u_{NC} + a_p u_p$$

$$+ a_{SC}u_{SC} + a_{NW}u_{NW} + a_{WC}u_{WC} + a_{SW}u_{SW} \quad , \tag{4.11}$$

where $a_{NE}, a_{EC}, \ldots$ are fractions which depend on the weightage that one wishes to give to each node of the element. These coefficients must all sum to unity, i.e.,

$$a_{NE} + a_{EC} + \ldots + a_{SW} = 1.0 \quad . \tag{4.12}$$

In the present investigation, the values of a_{NE}, a_{EC}, a_{SE}, a_{NC}, a_p, a_{SC}, a_{NW}, a_{WC} and a_{SW} used were 1/36, 4/36, 1/36, 4/36, 16/36, 4/36, 1/36, 4/36 and 1/36, respectively. The average velocity obtained is equivalent to the integral average of u over the element when u is approximately fitted with a second degree polynomial in x and y passing through the nine nodal values in the element.

Now,

$$A = 1/2 * RE * u_{AV} ,$$

and

$$B = 1/2 * RE * v_{AV} ,$$

are calculated. With these values of A and B, the x-momentum equation is solved for u_1 and the y-momentum equation is solved for v_1 using the algebraic equation (3.52). The coefficients in this equation are calculated by the subroutines 'HOMOG' and 'NHOMOG' given in Appendix D.

STEP 3. After calculating u_1 and v_1, the continuity equation is first used to calculate the velocity v_2 corresponding to u_1. Similarly, using the velocity v_1, u_2 is obtained from the continuity equation. According to the momentum averaging scheme, the velocities u_1 and u_2 along with v_1 and v_2 are averaged using the weighting factors A and B, i.e.,

$$u = \frac{u_1 A^n + U_2 B^n}{A^n + B^n}$$

and

$$v = \frac{v_1 A^n + V_2 B^n}{A^n + B^n} \qquad\qquad n \geq 1$$

Using the new values of u and v, A and B are again calculated in each element and u and v obtained. This process is repeated until a convergence of 10^{-3} is achieved, i.e., the maximum difference in the values of u or v (at any node) between two successive internal iterations becomes less than 10^{-3}.

After the convergence for u and v is achieved, the old values of u and v are replaced by the new values. Using these new values, the pressure is calculated once again. This procedure is repeated until the solutions for p, u and v converge. In the present investigation this numerical procedure seems always to produce stable solution. Thus, no under-relaxation is needed in the calculation.

CHAPTER 5

STAGNATION POINT FLOW

In Chapter 4, the method of numerical computation was discussed in detail. In this Chapter, the momentum averaging scheme is used to check separately the 9-point FA formula for pressure equation and the 9-point FA formula for momentum equation. The stagnation point flow is used for the purpose. The reason for selecting the stagnation point flow is that the exact analytic solution is available which can serve as a good comparison with the FA solution obtained here.

5.1 Verification of FA Solution for Momentum Equation

In this section, the FA solutions of the momentum equation (3.47) and continuity equation (3.52) are isolated for verification of accuracy and stability. This is done by substituting the known pressure distribution in the momentum equation so that only the continuity equation and momentum equations in x and y components are solved numerically by the momentum averaging scheme. This scheme stipulates that the u and v velocity components in each finite element can be approximately solved from the continuity equation and the two momentum equations and an average value taken in the finite element for each of the two velocities. This is a new scheme and deviates from the existing scheme [3] which calculates both u and v components from both the momentum equations and uses the continuity equation only for correction of pressure distribution in the pressure equation.

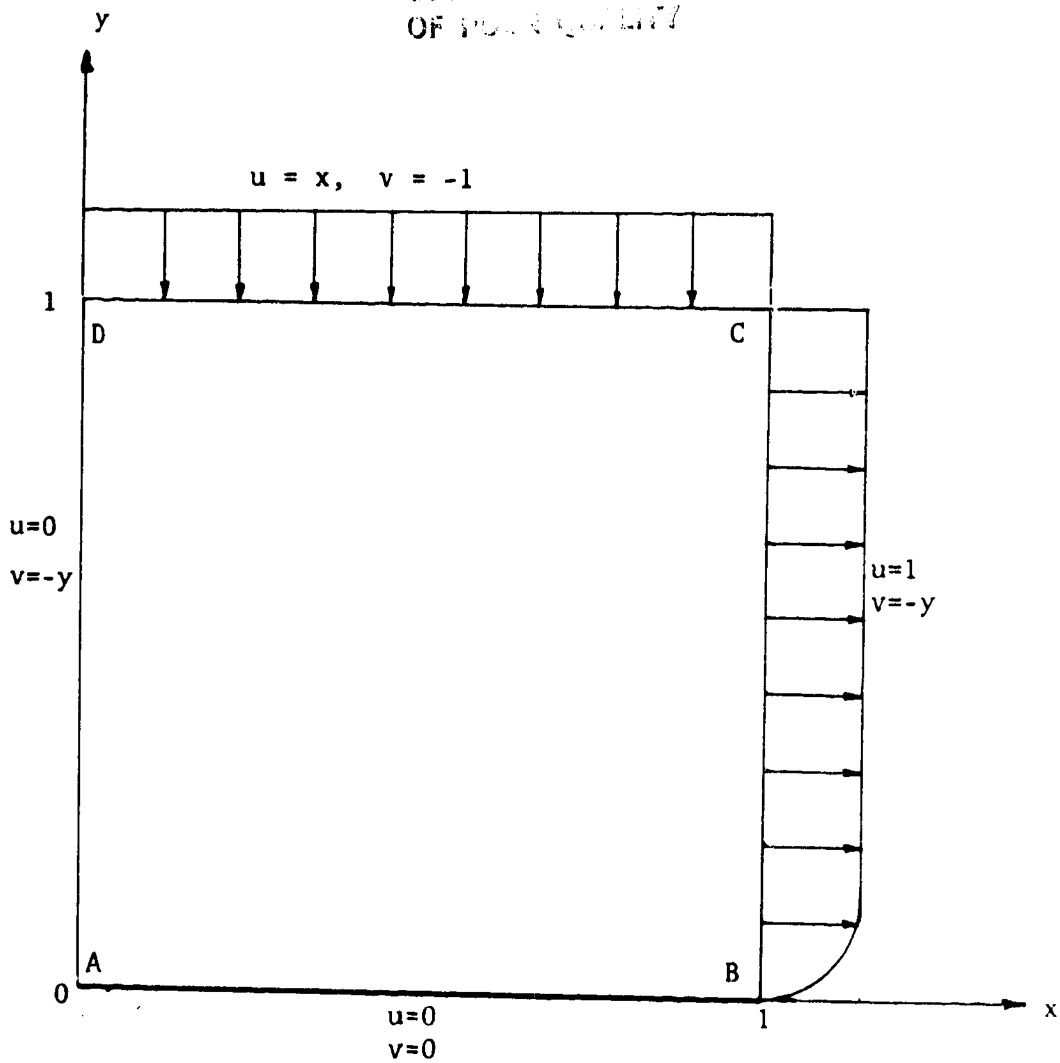

Figure 5.1 Stagnation Flow Problem

The computational domain for the calculation is shown in fig. (5.1). The plate is of unit length and the domain extends a unit distance in the normal direction. The origin and coordinate system are also shown in the figure. The boundary conditions away from the viscous layer near the wall are derived from the inviscid flow solution, namely $u=x$ and $v = -y$. However, on the side BC, the u velocity is given a near Blasius profile and $v = -y$. On the side CD, $u = x$ and $v = -1$. Between D and A, $u = 0$ and $v = -y$. On the surface of the plate, the no-slip boundary condition is used, i.e., $u = 0$ and $v = 0$.

From the potential flow analysis, it is known that the pressure at any point (x,y) in the domain is given by

$$p = -0.5(x^2 + y^2) . \tag{5.1}$$

Reynolds numbers of 100 and 400 are considered in the calculation. With the pressure distribution known, the 9-point FA formula for the momentum equation derived in Chapter 3 is used to calculate the velocity of the u- and v-components; and the corresponding components are computed by the continuity equation. The result is shown in fig. (5.2) for Re = 100. It is seen that the computed result outside the boundary layer is in agreement with the exact solution upto the fourth decimal place. The calculation is repeated for Re = 400 and the result given in fig (5.3) is again in good agreement with the exact solution.

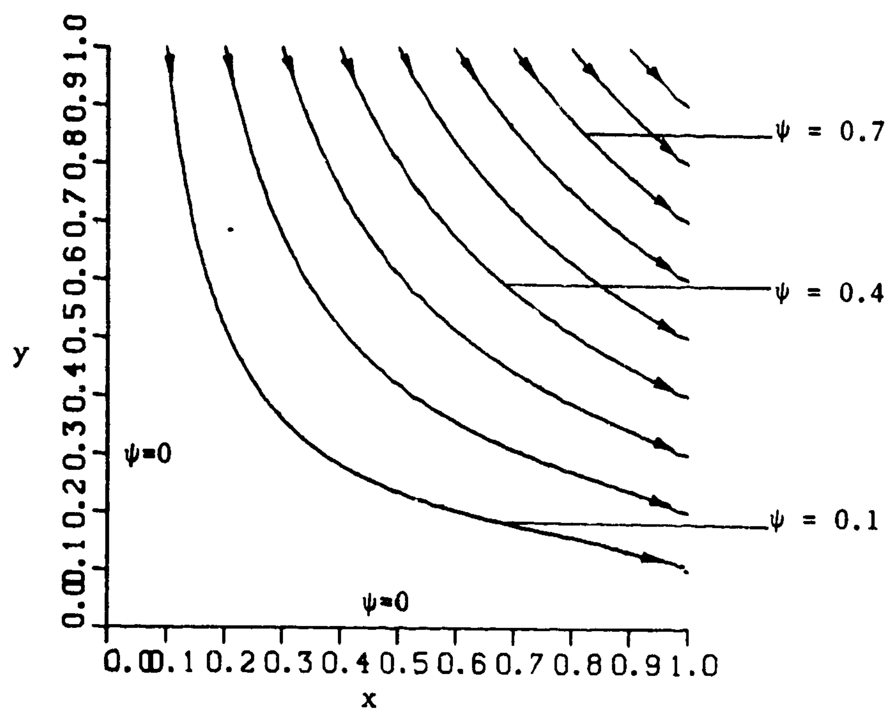

Figure 5.2 Streamlines for Stagnation Point Flow (Re = 100)

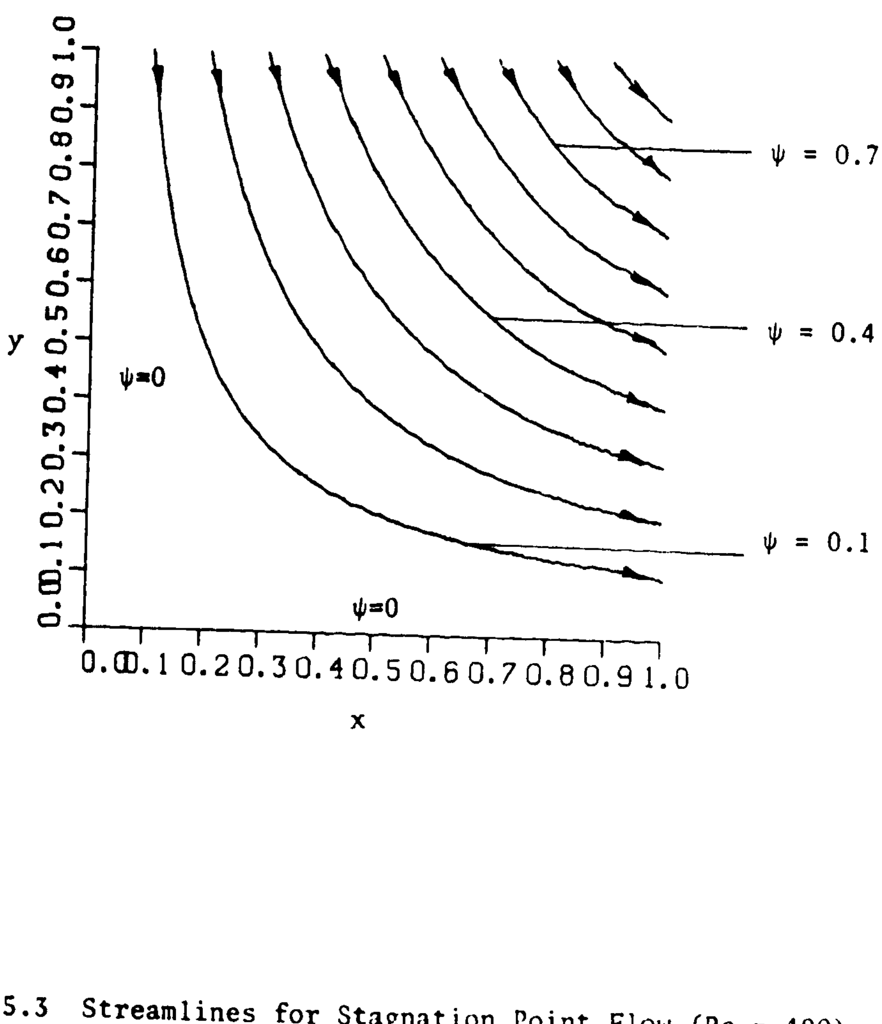

Figure 5.3 Streamlines for Stagnation Point Flow (Re = 400)

Table 5.1

Comparison of Exact and Calculated Values of Pressure

Location x	Location y	Exact Value	Calculated Value
0.1	0.1	-0.01000	-0.01017
0.1	0.3	-0.05000	-0.05016
0.1	0.5	-0.13000	-0.13016
0.1	0.7	-0.25000	-0.25016
0.3	0.1	-0.05000	-0.05016
0.3	0.3	-0.09000	-0-09015
0.3	0.5	-0.17000	-0.17014
0.3	0.7	-0.29000	-0.29015
0.5	0.3	-0.17000	-0.17014
0.5	0.5	-0.25000	-0.25014
0.5	0.7	-0.37000	-0.37014
0.5	0.9	-0.53000	-0.53016
0.7	0.3	-0.29000	-0.29015
0.7	0.5	-0.37000	-0.37014
0.7	0.7	-0.49000	-0.49015
0.7	0.9	-0.65000	-0.65016
0.9	0.3	-0.45000	-0.45016
0.9	0.5	-0.53000	-0.53016
0.9	0.7	-0.65000	-0.65016
0.9	0.9	-0.81000	-0.81017

5.2 Verification of the FA Solution for Poisson Equation

Having checked the FA numerical solution for the momentum and continuity equations, the Poisson equation is checked in a similar way. The velocity distribution in the domain is now given by

$$u = x \text{ and } v = -y. \tag{5.2}$$

In addition to the velocity distribution, the no-slip velocity is used at the plate surface. The 9-point FA formula for the Poisson equation (3.25) is now used to calculate the pressure in the domain. The result is shown in Table 5.1 where the computed and exact values are presented. The computed value of pressure is within 2% of the exact value. The error is probably from the truncation error of the finite difference approximation used in evaluating the velocity gradients which appear in the nonhomogeneous turn of Poisson equation.

With these two checks, the calculations for flow over a flat plate and wake are presented in Chapter 6 and the flow in a square driven cavity is discussed in Chapter 7.

CHAPTER 6

FLOW OVER A FINITE FLAT PLATE

6.1 Description of the Problem

In this chapter, the FA solution to the Navier-Stokes equations is used to calculate the velocity profile over and behind a finite flat plate. The Reynolds numbers used in the calculation are 100, 400 and 800 based on the plate length and free stream velocity. All previous analytic studies of wake calculation have been reported for large Reynolds numbers using boundary layer equations and often with additional assumption that the velocity difference in the wake is small compared with free stream velocity [5]. Furthermore, the near wake solution and the combined plate-wake solution are difficult to solve analytically because even when the Reynolds number is large, the boundary layer equation near the trailing edge is not valid as mentioned by Plotkin and Flugge-Lotz [6]. The full Navier-Stokes equations have to be used. In this chapter, the flow over a finite plate including the wake region is solved from moderate to high Reynolds numbers. Results for Re = 100, 400 and 800 are compared with the existing results of near wake and far wake solutions.

The computational domain for plate-wake region under consideration is shown in fig. (6.1). The plate is of length L and the computational domain in the wake region extends to a distance of 3L behind the plate in the direction of flow and a distance of L in the normal direction.

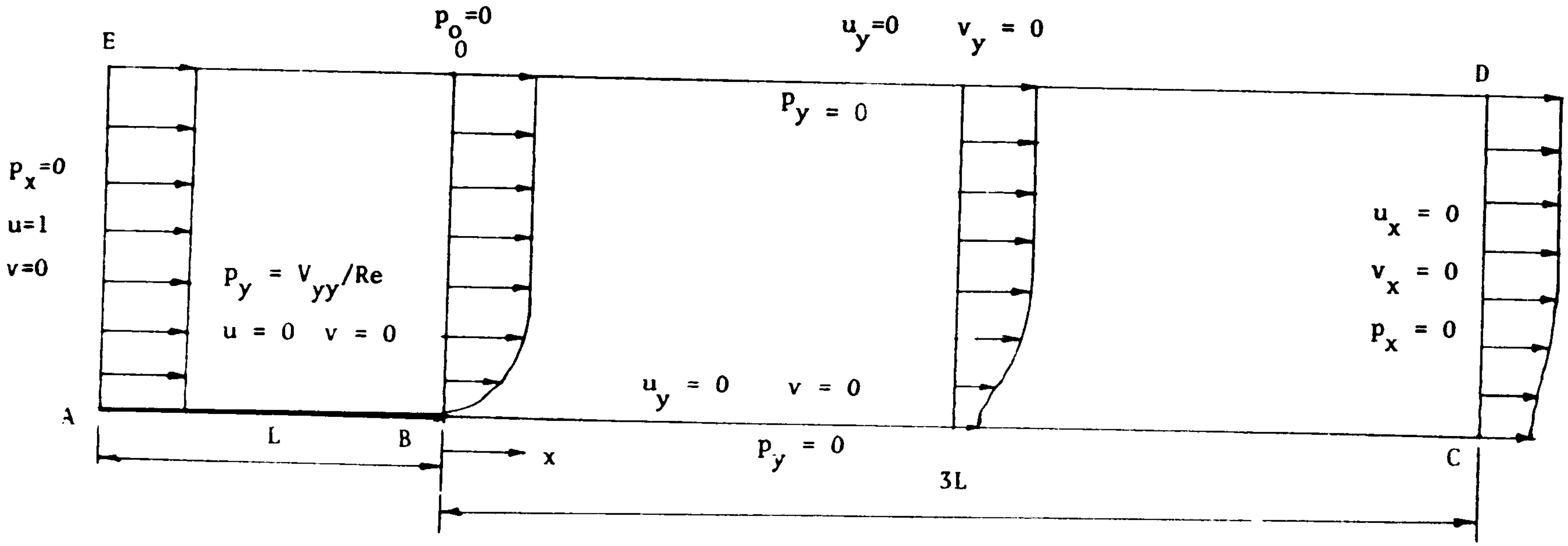

Figure 6.1 Flow Over a Flat Plate and in a Wake

The grid sizes in the x- and y-directions are selected depending on the Reynolds numbers. For Re = 100, the grid size in both the directions is 0.05. The reason for selecting different grid sizes for different Reynolds numbers is to ensure the boundary layer phenomenon in the flow field is accounted for. Since the boundary layer thickness may be estimated to be inversely proportional to the square root of the Reynolds number, it is necessary to have at least one node inside the boundary layer. Hence, the grid size should be at least equal to or smaller than $(1/Re)^{1/2}$.

It should be remarked that in the present calculation the full Navier-Stokes equations are used. Therefore, the calculation is not restricted to boundary layer phenomenon or large Reynolds numbers. Since the Navier-Stokes equations are elliptic partial differential equations, the boundary conditions must be given on all sides of the computational domain. It is assumed that the side DC in fig. (6.1) is far downstream from the flat plate and so the velocity varies slowly in the x-direction. Therefore, the downstream boundary conditions are taken as $u_x = 0$ and $v_x = 0$. If it were not for the economy of the computation, the DC boundary should be chosen further downstream, say at x = 10L or larger. From B to C it is assumed that the velocity profile is symmetric about the x-axis or $u_y = 0$ and $v = 0$. Along the flat plate AB, the no-slip boundary conditions are used i.e., u = 0 and v = 0. On the upstream side EA, the u-velocity is taken to be uniform and the v-velocity is zero, i.e., u = 1 and v = 0. As for the boundary condition on the side DE, the u- and v-velocity components are assumed to be constant in the y-direction as they are far away from the boundary layer or $u_y = 0$ and

$v_y = 0$. The pressure boundary conditions are as follows:

Side EA: $p_x = 0$

Side AB: $p_y = \frac{1}{Re} v_{yy}$

Side BC: $p_y = 0$

Side DC: $p_x = 0$

Side DE: $p_y = 0$

at point 0: $p = 0$

The computational procedures are described in Chapter 4. The number of overall iterations required for convergence of this solution is about 25 for Re = 400. As for the internal iterations, 10 iterations are needed for the convergence of the Poisson equation and about 20 for the momentum equation. For different Reynolds numbers, the number of iterations for convergence is increased. The numerical results are discussed below.

6.2 Discussion of Far Wake Solutions

In fig. (6.2), results are shown for Re = 400. The curves are for the u-velocity at distances of 0, 0.5L, L, 1.5L, 2L, 2.5L and 3L from the trailing edge of the flat plate. These results are compared with those of Tollmein [7] as shown in fig. (6.3). Although Tollmein gave results for $x < 3L$, he stated that the results in fig. (6.3) are valid only at a distance greater than 3L from the trailing edge of the plate and for large Reynolds numbers. This is because in obtaining his result, Tollmein uses a Blasius profile at the trailing edge as the

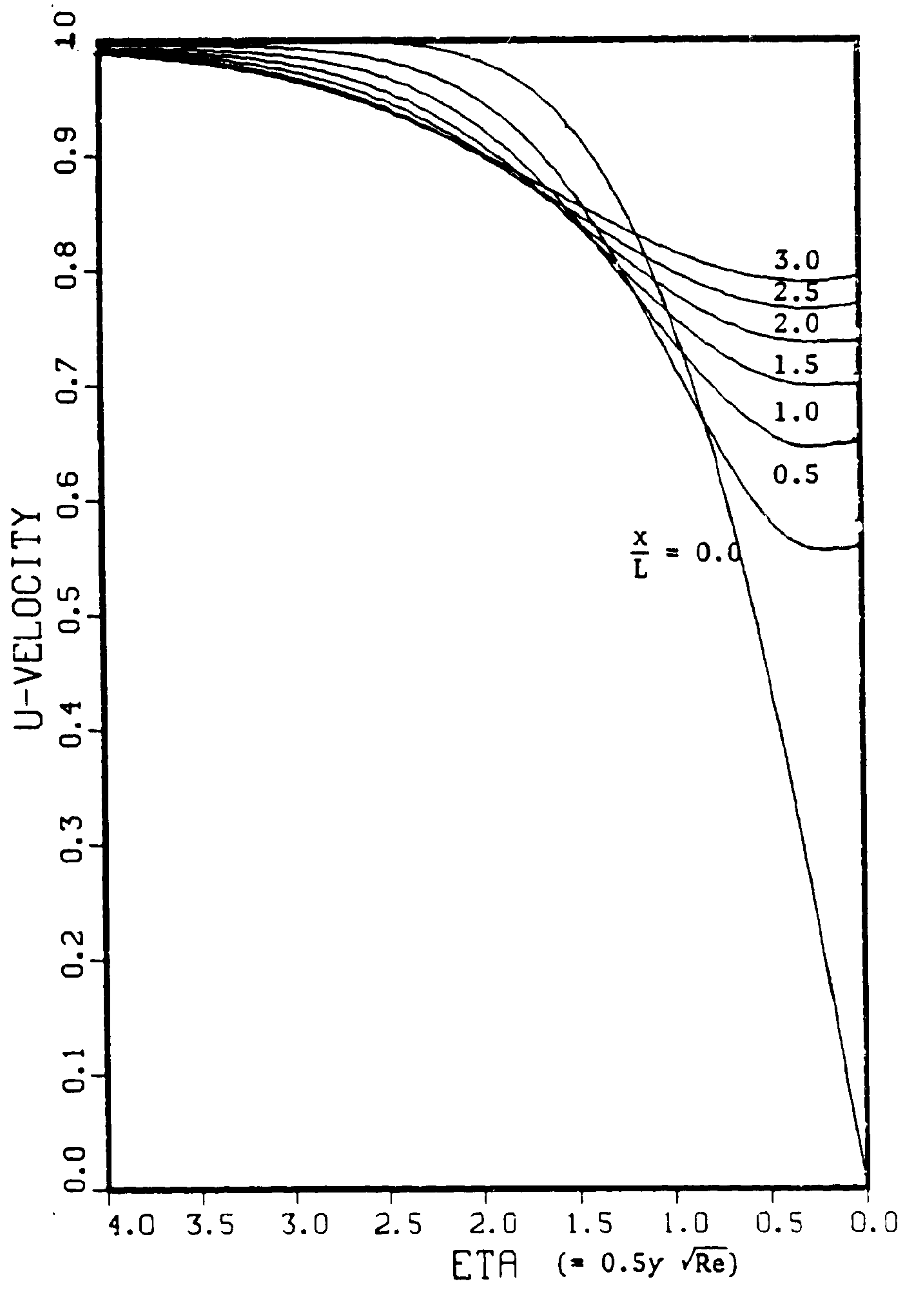

Figure 6.2 Velocity profiles in a Wake for Re = 400

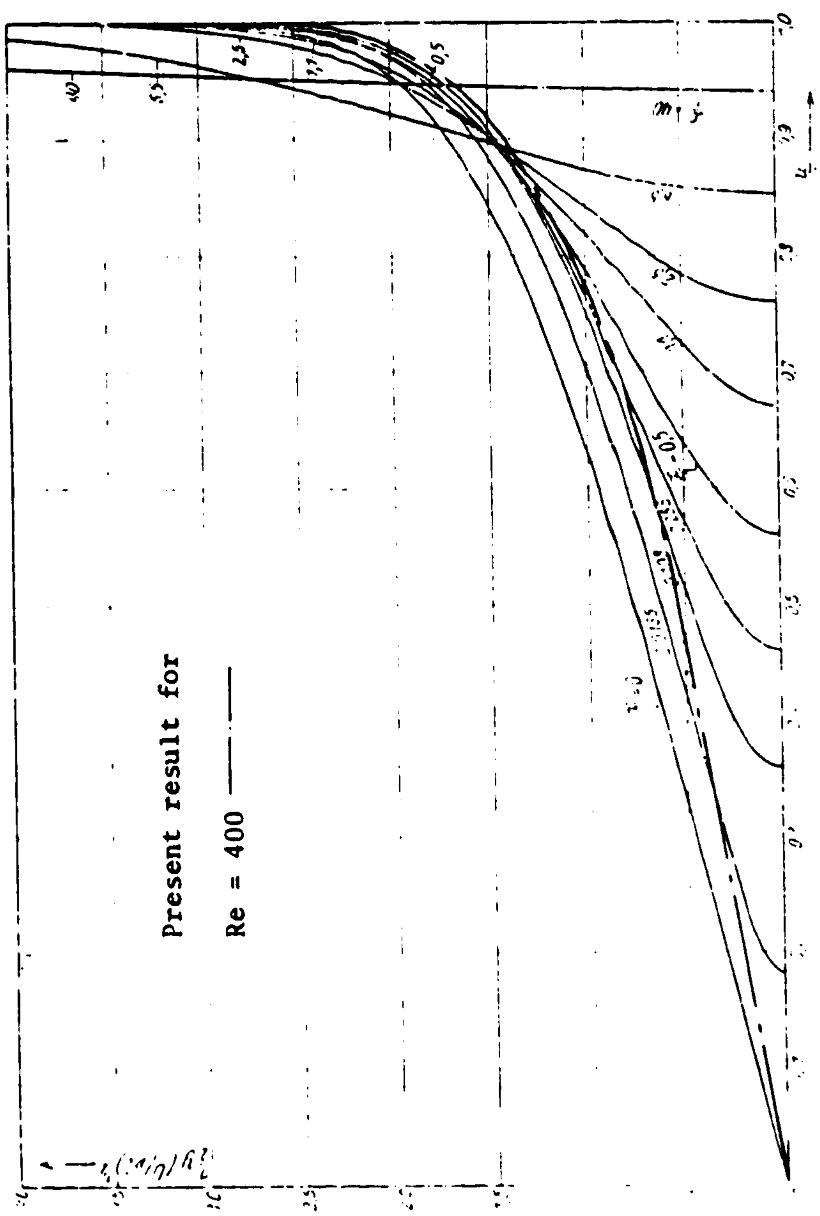

Figure 6.3 Tollmein's Velocity Profiles in a Wake for Large Re

boundary condition. Since the Blasius profile requires the assumption of large Reynolds numbers his result is valid only for Reynolds numbers greater than 10^3. Further, Tollmein used the boundary layer equations instead of Navier-Stokes equations in calculating the flow behind the trailing edge. He, in addition, simplified the boundary layer equations by assuming that the velocity defect in the wake is small compared to the main stream velocity. These assumptions are likely to cause substantial error in the solution in the near wake region where the velocity defect is still large as can be seen from fig. (6.3). Even for u as large as 2.5L, the velocity defect in fig. (6.3) at the centerline is 0.25 which is 25 percent of the main stream velocity. Therefore, the assumption that the velocity defect is small is invalid in the range $0 \leq x \leq 2.5L$.

In the present study, the full Navier-Stokes equations are used to solve the flow over and the wake behind a flat plate. The momentum equation used here is

$$uu_x + vu_y = \frac{1}{Re}(u_{xx} + u_{yy}) - p_x\ , \qquad (6.1)$$

as compared to the equation

$$u_{1x} = \frac{1}{Re}\ u_{1yy} \qquad (6.2)$$

used by Tollmein. In equation (6.2), u_1 is the velocity defect i.e., $u_1 = 1-u$. Since the present method uses the full Navier-Stokes equations, it can be, theoretically speaking, used to calculate velocity profiles for Reynolds numbers ranging from very small to large values. Further, the combined boundary layer-wake solution can be obtained by this method without specifying the velocity profile at the end of the plate. One of the purposes of calculating the flow over the finite flat plate is that there is no existing work which gives correct solutions to the problem with the boundary layer-wake interaction at moderate Re. Since there is no exact solution available to verify the accuracy of the present results, the computational domain is extended to a distance of 3L behind the trailing edge so that it may be approximately compared with Tollmein's far wake result which is claimed to be valid for $x > 3L$. It is expected that the solution obtained by using boundary layer equation must fail at a distance of $O(Re^{-\frac{1}{2}})$ from the trailing edge because at this distance x is of the same order as y (the normal distance from the centerline) and the full Navier-Stokes equations must be used.

Consider now fig. (6.2) for Re = 400 and fig. (6.3). Fig. (6.3) is restricted to far wake region or $x > 3L$ and large Re. Since fig. (6.3) does not give a curve for $x = 3L$, the velocity profile for $x = 3L$ is approximately interpolated between the curves for $x = 2.5L$ and $x = 6.5L$. Comparing Tollmein's result at $x = 3L$ for large Re with the present result for $x = 3L$ at Re = 400 in fig. (6.2), it is seen that the axial velocity at a given x in fig. (6.2) is larger than that in fig. (6.3). The velocity predicted in the present analysis is not a Blasius profile at the trailing edge while the velocity profile in fig. (6.3) is a Blasius profile. At any x location the centerline velocity in fig. (6.3) asymptotically increases from a smaller value to a value greater than that in fig. (6.2). Physically this implies that the entrainment rate is different in the two cases. The reason for this difference in entrainment rate can be given by considering the momentum equation in the x-direction

$$uu_x + vu_y - (u_{xx} + u_{yy})/Re \qquad (6.5)$$

Denoting the velocity difference u_1 as

$$u_1 = u_\infty - u \tag{6.4}$$

and substituting into equation (6.3) yields

$$(u_\infty - u_1)\,(u_\infty - u_1)_x + v(u_\infty - u_1)_y = \frac{1}{Re}\,[(u_\infty - u_1)_{yy} + (u_\infty - u_1)_{xx})] \tag{6.5}$$

Since u_∞ is a constant (i.e., $u_\infty = 1$), the above equation on simplification becomes

$$u_\infty u_{1x} - u_1 u_{1x} + v u_{1y} = (v u_{1yy} + u_{1xx})\,\frac{1}{Re}\,, \tag{6.6}$$

$$u_\infty u_{1x} - u_{1yy}/Re = u_1 u_{1x} - v u_{1y} + u_{1xx}/Re\ .$$

The equation used by Tollmein was

$$u_\infty u_{1x} - u_{1yy}/Re = 0\ . \tag{6.7}$$

So the terms on the right hand side of equation (6.6) are not present in equation (6.7). The effect of these terms can be neglected only when u_y and its gradient are small and when $x > 3L$. Therefore, the difference between the present calculation based on the full Navier-Stokes equations and Tollmein's calculation based on equation (6.7) create the difference in the entrainment rate. This difference manifests in the different speed of recovery of the velocity defect particularly in the near wake region. It should be noted that since Tollmeins' approximation and entrainment prediction in the near wake region are not valid, the result at far wake region, even though profile is approximately correct, requires a shift in the origin of the trailing edge to account for the defect in the entrainment rate.

The results for Re = 800 are shown in fig. (6.4). These results seem to have better similarity with the Blasius solution at the trailing edge and with other existing solutions at a distance x > 3L. However the difference is still appreciable. Again, the use of equation (6.7) by Tollmein in calculating is one of the reasons. Further, Tollmain's results were obtained, with very large Reynolds numbers which should be in the order of 10^4 to 10^6. Hence, the reasons for the discrepancy between this result and other results are the same as those discussed for Re = 400.

6.3 Discussion of Near Wake Solution

Regarding other previous near-wake solution, Goldstein [8] in 1930 first calculated the flow downstream of the trailing edge of a thin flat plate at zero incidence. He used the boundary layer equations assuming that the Reynolds number is very large. Hence he appropriately took the Blasius profile as the boundary condition at the trailing edge. However, Goldstein [8] solved the boundary layer equation in the wake with a series solution for the velocity profile by expanding the series from the trailing edge. The solution is thus valid only for small distances from the trailing edge. As the distance from the axis increases the result becomes inaccurate. Further, Goldstein's solution has an algebraic singularity at the trailing edge of the plate. Like Tollmein's result, his result is also valid only for large Re. Fig. 6.5 shows Goldstein's result as compared to the present result for Re = 400 and 800 in figs. (6.2) and (6.4) in the near wake of a flat plate. Goldstein's axial velocity at x = 0.2L is 0.43 as compared to 0.45 for Re = 800 in the present study.

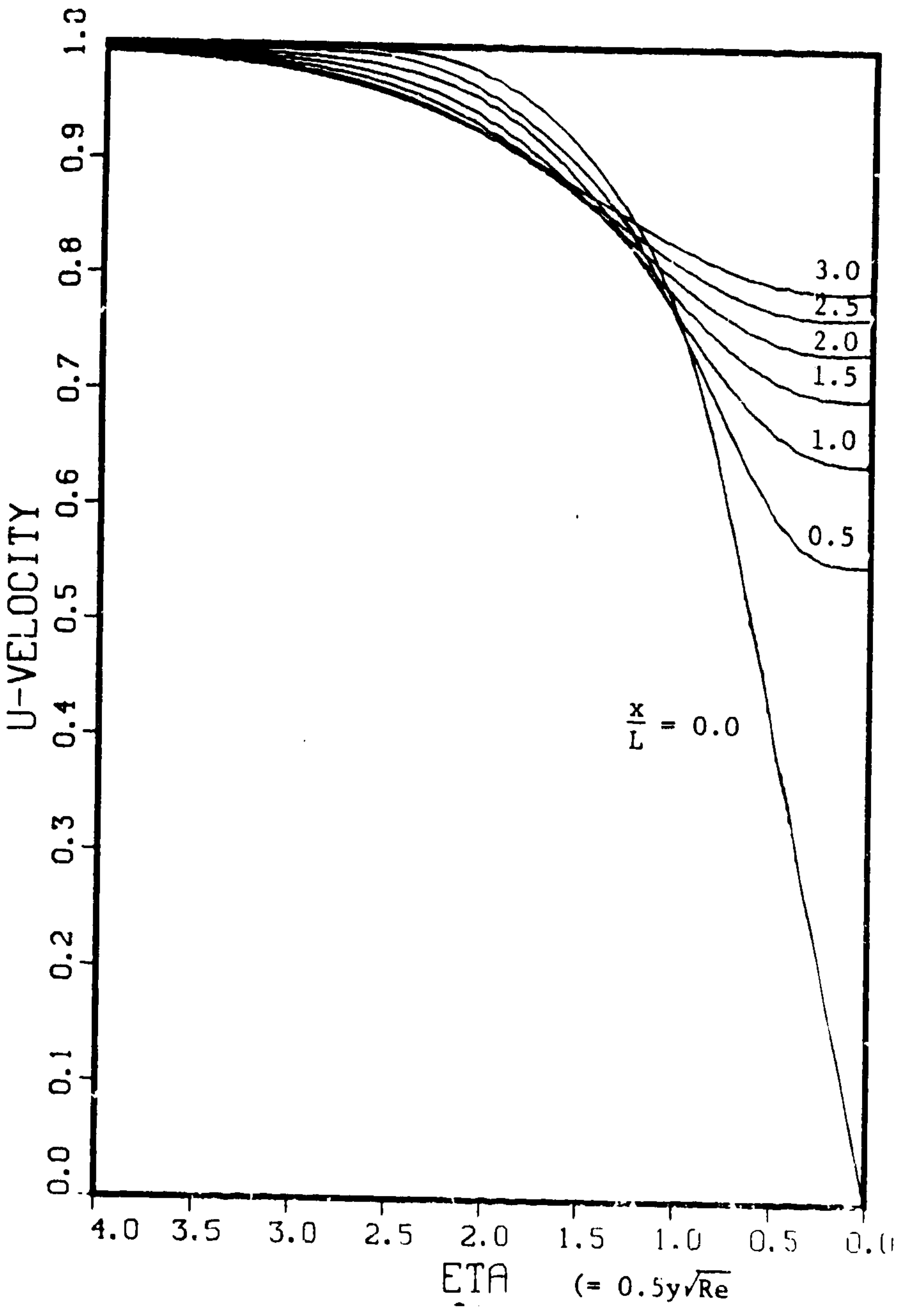

Figure 6.4 Velocity Profiles in a Wake for Re = 800

Since the result obtained by Goldstein in fig. (6.5) is from the boundary layer equation, which is parabolic, there is no mechanism in his analysis by which the wake solution behind the trailing edge can be communicated to the flow upstream of the plate. This is true only for large Reynolds number and the flow is governed by boundary layer equation. However, it was pointed out by Plotkin and Flugge-Lotz [6] that no matter how high the Reynolds number, there must exist a region near the trailing edge in which the boundary layer assumptions are not valid, and the full Navier Stokes equations must be used. In short, the present calculation differs from Goldstein's calculation in that the Reynolds numbers are 400 and 800 instead of large Reynolds numbers and that the full Navier-Stokes equations are used instead of boundary layer equation. These are possibly the reasons for the difference between Goldstein's results in fig. (6.5) and the present result for Re = 400 and 800.

Fig. (6.6) also compares the present calculation with Goldstein's result for the axial velocity for x ranging from 0 to 0.6. The chain-dot line obtained by the present method for Re = 100, 400 and 800 are shown in the same figure. The axial velocity obtained by Goldstein is slightly less. This difference is probably because his calculation was not based on the full Navier Stokes equations.

Plotkin and Flugge-Lotz [6] had solved by finite difference method the flow over a finite plate and wake. They divided the region into two parts. In the region closer to the trailing edge they calculated the full Navier-Stokes equations and in the other region far downstream, they used the boundary layer equations. Unfortunately, they too, like others,

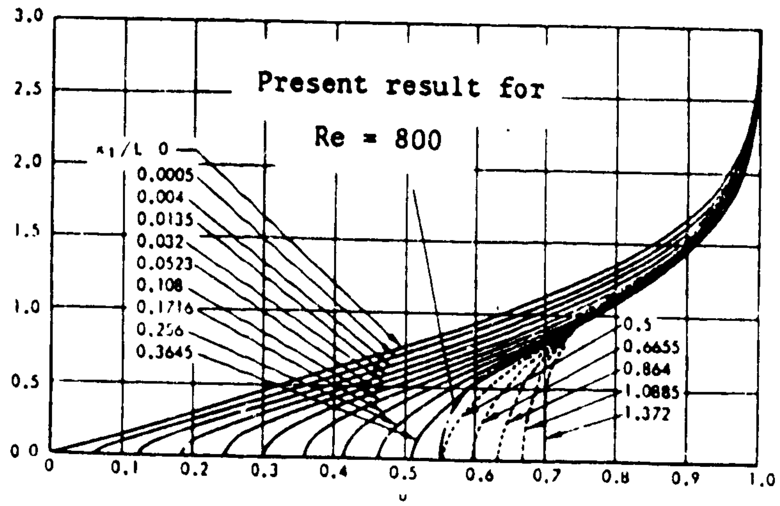

Figure 6.5 Goldstein's Velocity Profiles in the Near Wake of a Flat Plate for Large Re

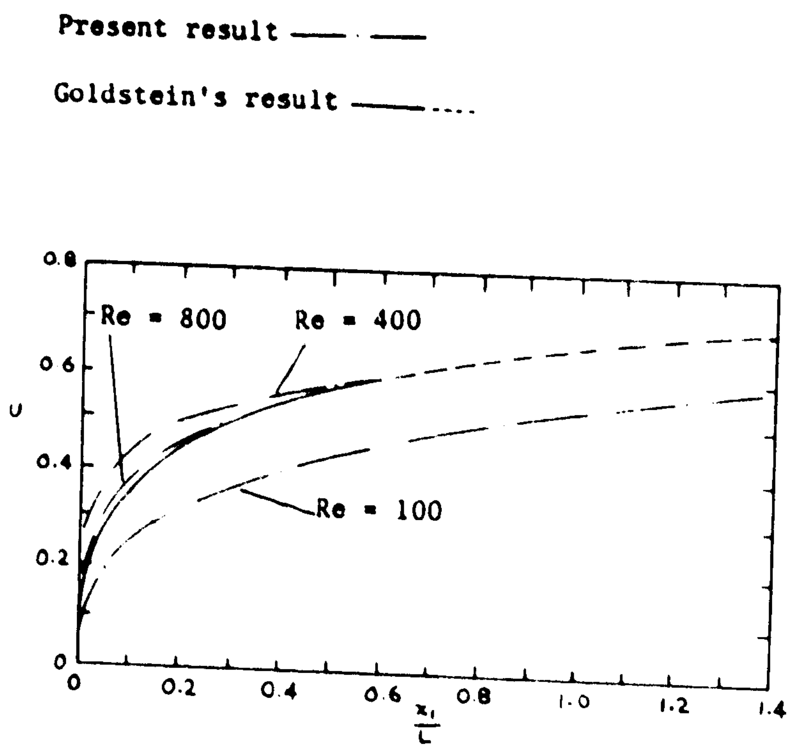

Figure 6.6 Goldstein's Axial Velocity Distribution in the Near Wake of a Flat Plate for Large Reynolds Numbers

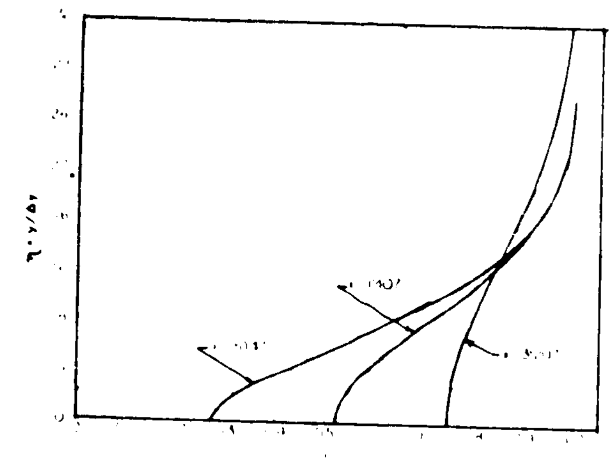

Figure 6.7 Plotkin and Flugge-Lotz Velocity Profiles in the Near Wake of a Flat Plate ($R = 10^6$; $\Delta x = 0.006$)

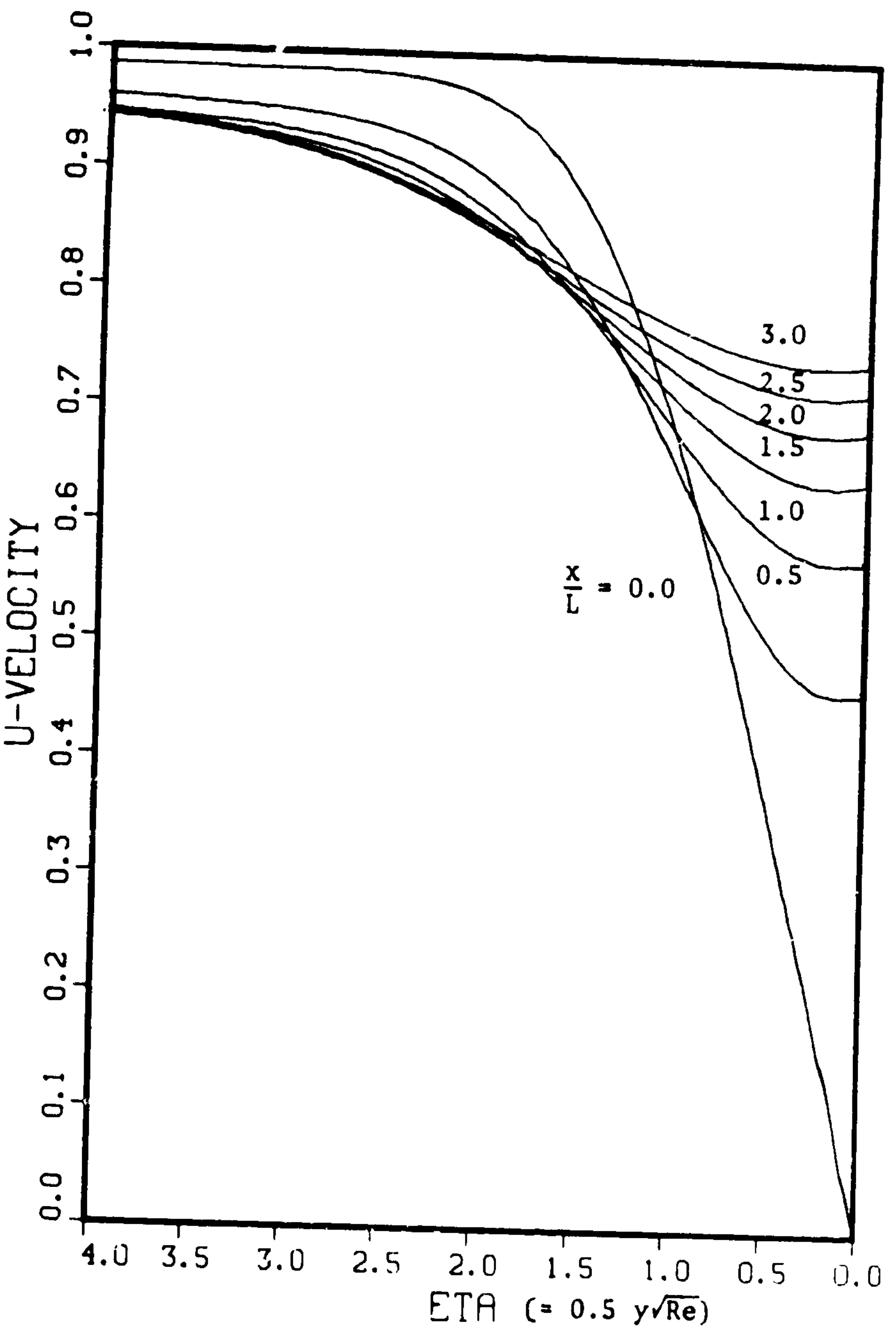

Figure 6.8 Velocity Profiles in a Wake for Re = 100

solved for Re greater than 10^5. They mentioned that the trailing edge disturbance necessitates solving the complete Navier-Stokes equation in the trailing edge region even at high Re. This implies that the flow variation in both x and y directions is important even for large values of Re. Their result is shown in fig. (6.7) which is again slightly different from the present result, the main reason being the large difference in Reynolds numbers.

The behaviour of the velocity profiles at Re = 100 is shown in fig. (6.8). These profiles are markedly different from those obtained by boundary layer equation. It should be mentioned that even though the flow is laminar over the flat plate for large Re, the flow may quickly become turbulent once it leaves the trailing edge of the plate. This is because the velocity profile in the wake has an inflection point, which according to Rayleigh inflection point theorem [9] of stability analysis could become unstable in the wake region. Therefore, the laminar solution in the near wake region for moderate Reynolds numbers is really of practical interests.

6.4 Conclusion

The flow over a finite flat plate investigated in the past is mainly for large Reynolds numbers. However, the same flow at moderate Reynolds numbers has given difficulty to investigators over the past few years. The main problem was that the full Navier-Stokes equations must be used. Therefore little was known about the plate-wake interaction. In most of the previous investigations the Blasius solution or boundary layer solutions has been used as a boundary condition at the trailing edge. Further, a boundary layer equation is used in the wake region. However, if the plate is finite and Reynolds number is moderate, the wake-flow may interact with the viscous flow over the plate. So the Blasius solution can no longer be assumed as the boundary condition for the wake region. In the present method, no such assumptions are made and the full Navier-Stokes equations are solved with the correct boundary conditions imposed on the leading edge rather than the trailing edge of the plate.

CHAPTER 7

FLOW IN A SQUARE DRIVEN CAVITY

The algebraic 9 point formulae derived under the FA method given in Chapter 3 were verified in Chapter 5 for their accuracy. These FA formulae were combined to form a numerical method of computation and used to solve the problem of flow over a finite flat plate in Chapter 6. Results for moderate Reynolds numbers were obtained and compared with the existing results for high Reynolds number with boundary layer assumption. Fairly accurate results are obtained from the present FA numerical solution of Navier-Stokes equation for primitive variable. However, the momentum averaging iterative scheme has not been rigorously tested as the flow over the finite plate is always dominated by the x-direction mementum. One, thus, may still have some doubt that the validity of the momentum averaging iterative scheme. In this chapter the problem of flow in a driven cavity is solved by the proposed momentum averaging scheme and the FA averaging method. One, therefore, expects that in solving the cavity flow, the momentum averaging iterative scheme is put on a rigorous test as the flow field of the problem contains recirculation and separation and the dominant momentum components rapidly shift from one finite element to the other in the flow field. Furthermore, the solution can be compared with results obtained by many other investigators [2,3,4].

Fig. (7.1) shows a cavity in which the fluid is driven by a plate AB moving at a velocity u = 1 from left to right. The calculations are done for Re = 100, 400 and 800. The boundary conditions for u and v on all the sides are zero except for the plate AB where u = 1. The pressure boundary condition cannot be specified exactly and so a Taylor series expansion of pressure about the four walls to their immediate interior neighboring nodes is carried out. Details of the derivation of the pressure boundary condition are given in Chapter 4. The choice of the grid size is based on the Reynolds number as is discussed in Chapter 6. In short, higher the Reynolds number, the finer is the grid. For Re = 100, the grid size (h) is 0.025 in both directions..For Re = 400, it is also 0.025, and for Re = 800, it is 0.0167.

7.1 Velocity Distribution

In fig. (7.2), the u-velocity profiles along a vertical line through the geometric center of the cavity are shown. The profiles are for Re = 100, 400 and 800. It is seen that at Re = 100, the velocity curve is smooth showing that the diffusion of viscous effects penetrates throughout the cavity but for larger Re the velocity gradient in most part of the cavity is constant and the boundary layer-like velocity profile is seen such that near the top and bottom sides of the cavity, the velocity gradient is very steep. This accounts for the fact that the shear stress of the skin friction is large for high Reynolds numbers. The maximum negative velocity increases and shifts toward the top surface of the cavity. For Re = 400, the maximum negative velocity is about 0.28 and for Re = 800 it is about 0.30. The result obtained

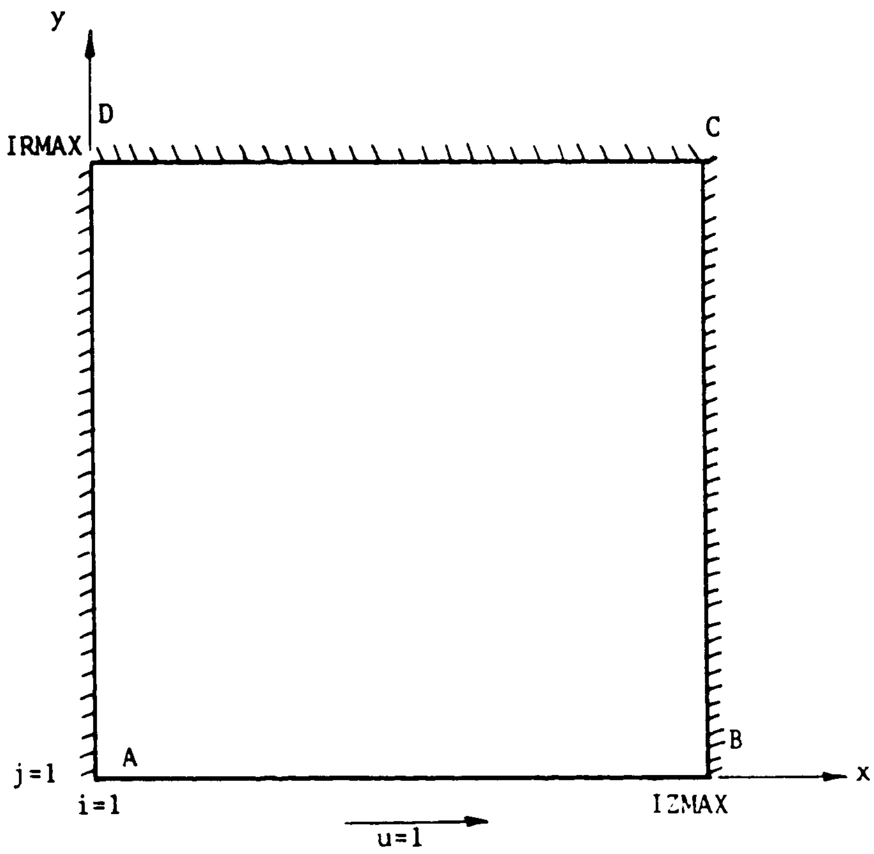

Figure 7.1 Flow in a Square Driven Cavity

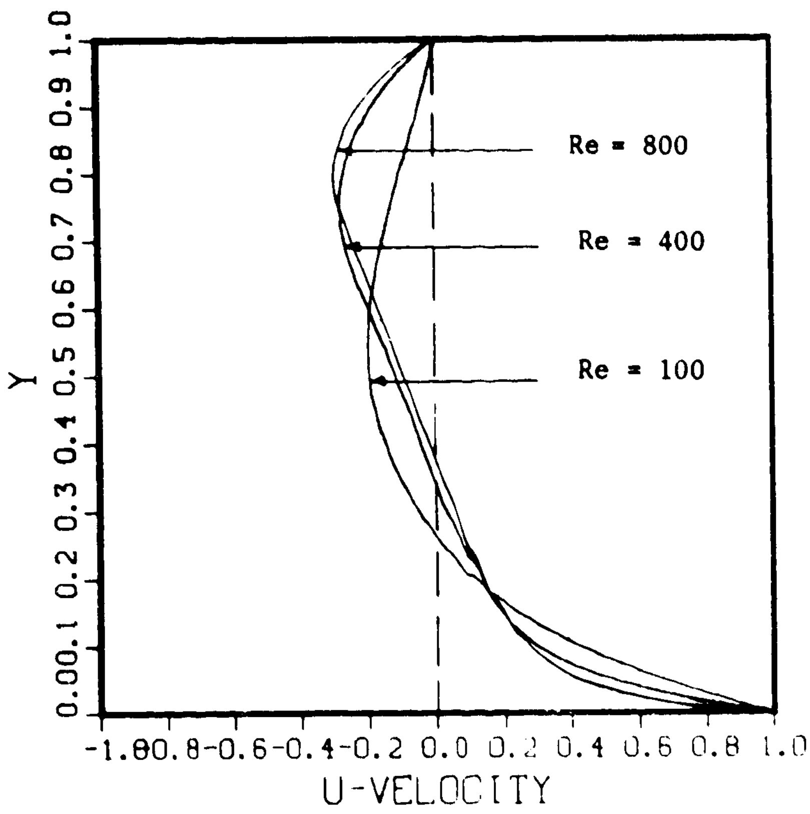

Figure 7.2 Velocity Profiles along a Vertical Line through the Geometric Center for Re = 100, 400 and 800

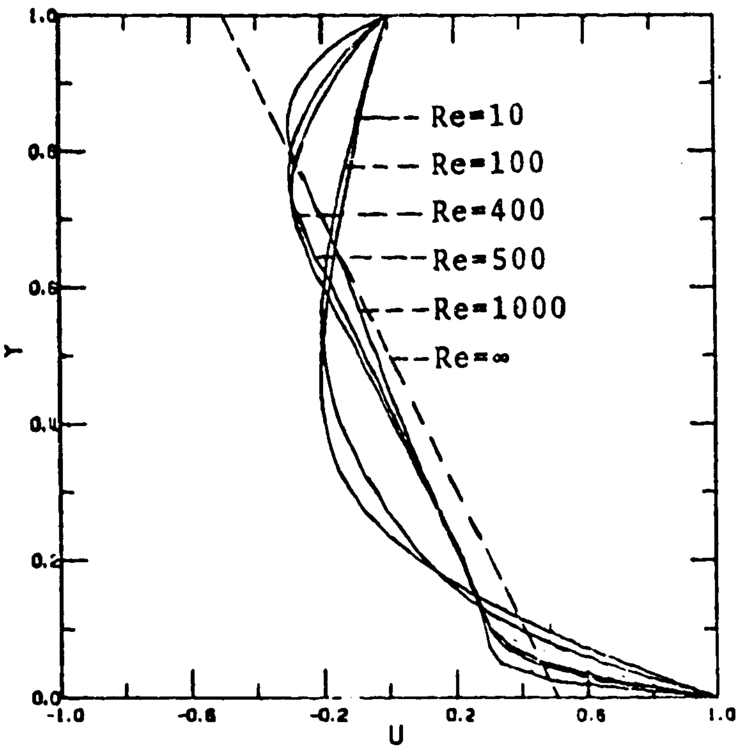

Figure 7.3 Velocity Profiles along a Vertical Line Through the Geometric Center for Different Re as Obtained by Chen et al.

by Chen and Naseri and Li [2] from vorticity-stream function formulation is shown in fig. (7.3). The two results are fairly the same. The fact that the two results agree with each other and that Chen et al's FA numerical solution does not utilize momentum averaging idea shows that the averaging iterative scheme is valid in this problem where the dominant momentum shifts widely from an element to an element.

It is obvious that, as the Reynolds number increases, a greater number of iterations are required for the convergence of the results. For Re = 100, the Poisson equation needed 10 internal iterations for convergence to 10^{-4} whereas the momentum equation needed 20 interval iterations to converge to 10^{-3}. For the combined solution of u, v, p to converge and stabilize , 25 overall iterations were given. For the case of Re = 400, the Poisson equation required 15 interval iterations, the momentum equation required 25 internal iterations and the numerical scheme required about 35 overall iterations for convergence. For Re = 800, the number of internal iterations for Poisson equation was about 20 and for momentum equation it was about 40. Overall, 45 iterations were needed for convergence. It was noted during these calculations that the number of internal iterations required for momentum equation to converge had a relation with the grid size. Indeed, the number of interval iterations needed was found to be in proportion to the number of grid points in a column. This is so since in the Gauss-Seidal iterative method one can expect that the substantial improvement in the nodal values in the first internal iteration will be confined to the first few rows. Therefore, it seems that, if there are 40 nodes in a column, then probably 40 iterations are required to obtain

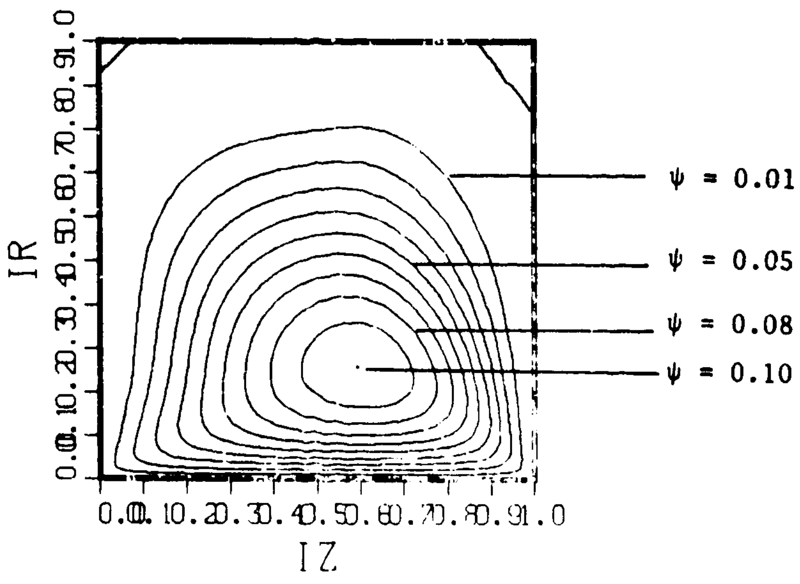

Figure 7.4 Streamlines for Re = 100

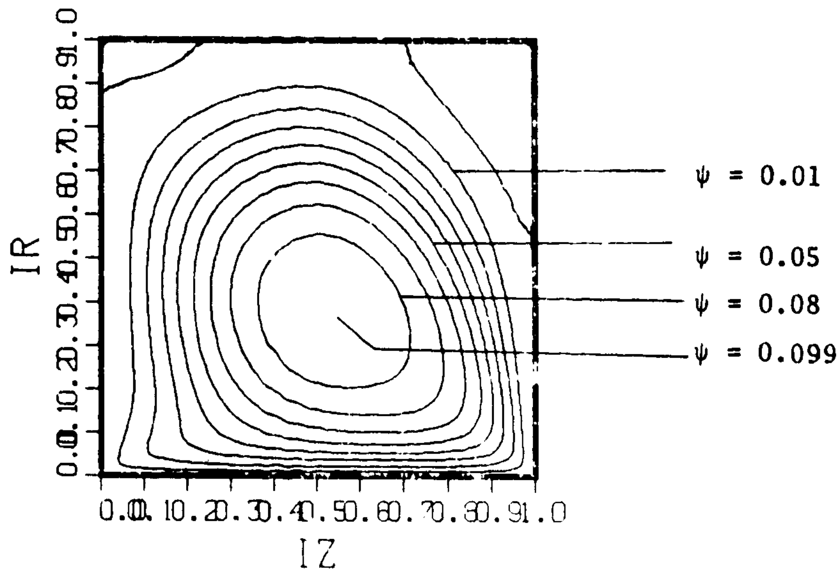

Figure 7.5 Streamlines for Re = 400

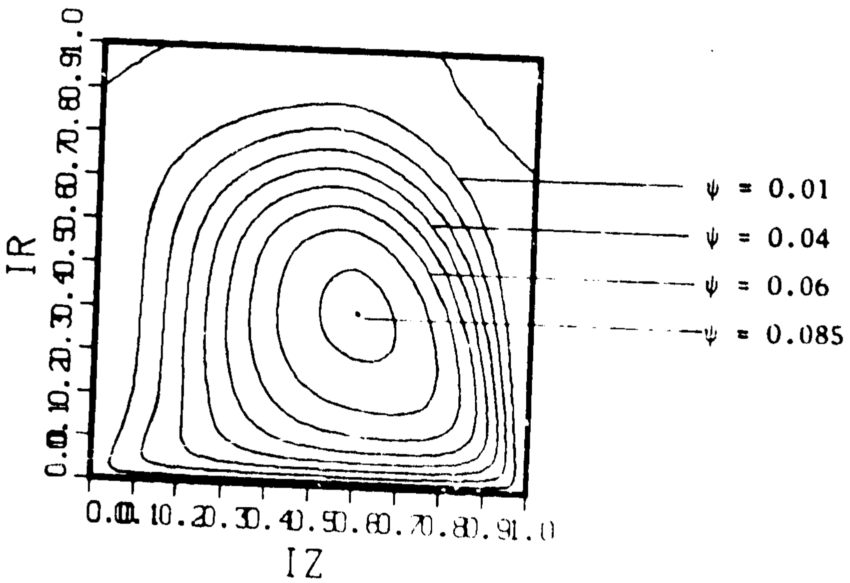

Figure 7.6 Streamlines for Re = 800

a good numerical solution. The quantitative values for typical nodes in the cavity for various Reynolds numbers are given in Table 7.1.

7.2 Streamline Pattern

Since the velocity distribution alone does not give a good idea of the flow pattern in the cavity, the stream function was computed from the velocity distribution. This is done by solving the equation

$$\psi_{xx} + \psi_{yy} = - (u_y - v_x) \tag{7.1}$$

Equation (7.1) is essentially a Poisson equation and is identical to the Poisson equation for pressure (3.4) in Chapter 3. Since the FA solution of Poisson equation is already given in equation (3.25), equation (7.1) is readily solved if the vorticity (u_y-v_x) is computed from the known FA solution of u, v by difference approximation. Fig. (7.4) gives the stream function contours for Re = 100. The contours for Re = 400 and Re = 800 are given in figs. (7.5) and (7.6). The stream function at the center of the vortex has a maximum value of 0.10 which compares fairly well with the result of 0.101. The streamlines for Re = 100 given in reference [2] are shown in fig. (7.7). It is noted that the separation at the two top corners are predicted from the present FA method for primitive valuables.

7.3 Pressure Distribution and Force Balance

Fig. (7.8) gives a plot of isobars in the cavity for Re = 100 and

Table 7.1

Comparison of Velocity at Various Points with that of Chen et al.

Location		u-velocity at different Reynolds Numbers			
		100		400	
x	y	[2]	Present	[2]	Present
0.5	0.1	0.40	0.42	0.31	0.30
0.5	0.2	0.09	0.11	0.20	0.16
0.5	0.3	-0.08	-0.05	0.11	0.05
0.5	0.4	-0.18	-0.15	0.00	-0.04
0.5	0.5	-0.21	-0.20	-0.10	-0.12
0.5	0.6	-0.18	-0.19	-0.20	-0.20
0.5	0.7	-0.14	-0.15	-0.27	-0.27
0.5	0.8	-0.09	-0.10	-0.27	-0.27
0.5	0.9	-0.05	-0.04	-0.18	-0.18

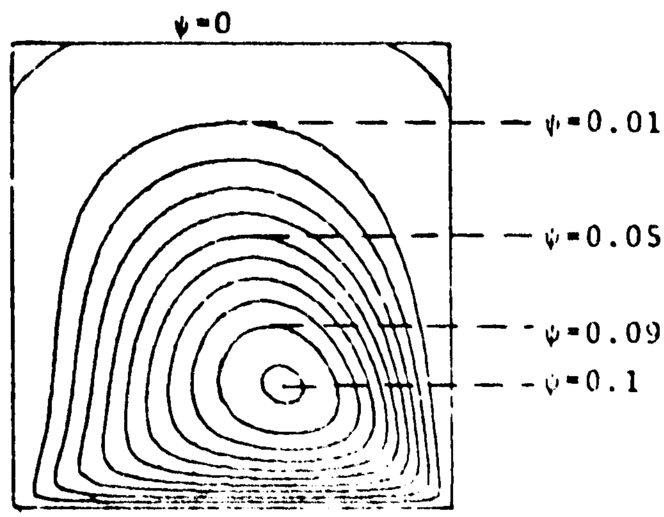

Figure 7.7 Streamlines for Re = 100 as Obtained by Chen et al.

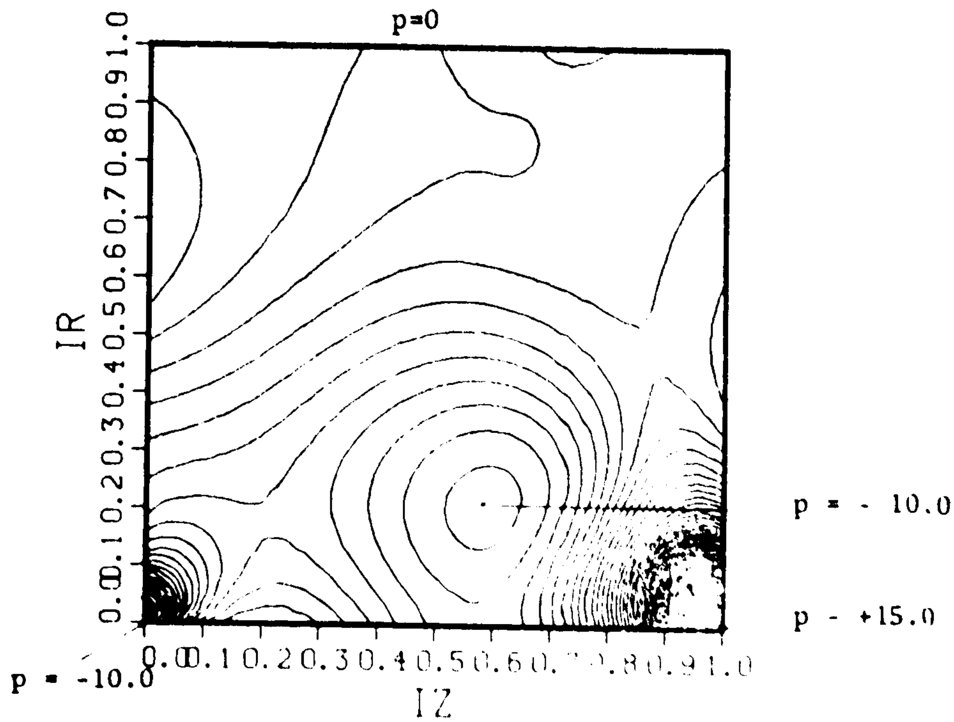

Figure 7.8 Pressure Distribution for Re = 100

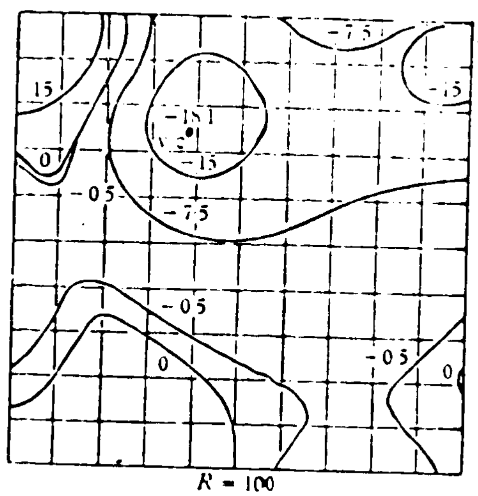

Figure 7.9. Pressure Distribution Obtained by Burggraf.

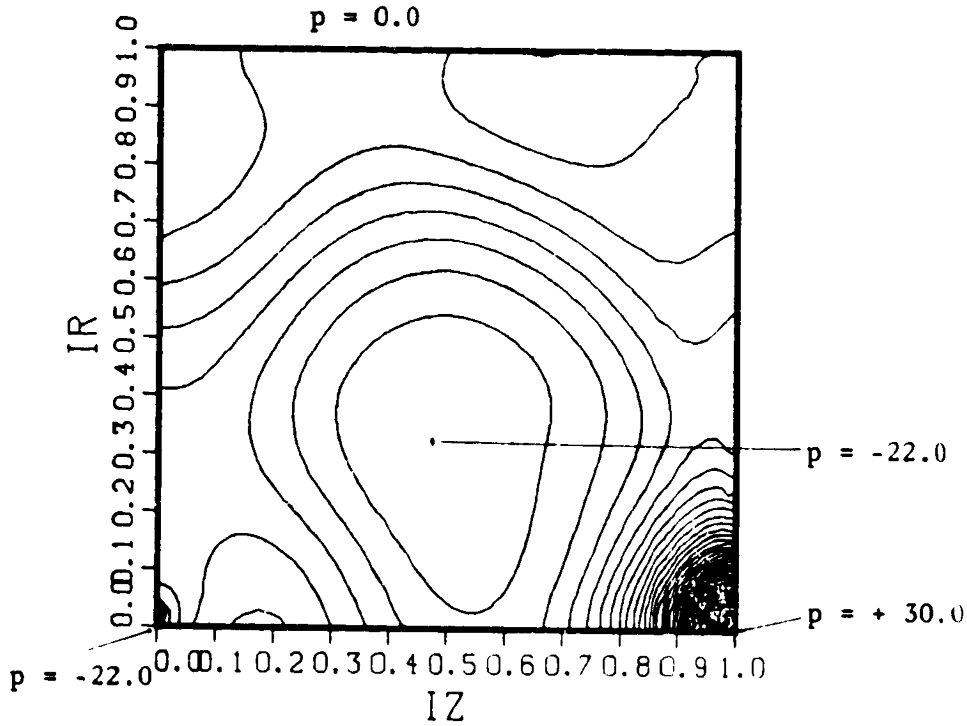

Figure 7.10. Pressure Distribution for Re = 400.

41 x 41 grid. The result is quite accurate. This result is quite similar to that in fig. (7.9) as obtained by Burggraf [4]. Another check for the pressure in the cavity was made by making a force balance. That is, the full cavity was taken as a control volume and the pressure force and viscous shear force in the x-direction and in the y-direction were separately added and it was found that they are very small. In the x-direction the net force in dimensionless units is of the order 10^{-3} and in the y-direction it is 10^{-4}. That is, the force balance is good upto the third decimal point. This is obtained for all the three different Reynolds numbers. Fig. (7.10) gives the pressure distribution for Re = 400. Comparison of fig. (7.8) and fig. (7.10) shows that as the Reynolds number increases from 100 to 400 the pressure difference between the two corners at the bottom increases from $\Delta p = 25$ to $\Delta p = 55$. It is interesting to see that the pressure difference between the vortex center and the downstream corner is also the same for the Reynolds numbers considered.

From the above discussion of the FA solution for flow in closed square cavity by momentum averaging scheme it may be concluded that the FA solution and the momentum averaging scheme to work very well in predicting the complex recirculating flow.

CHAPTER 8

CONCLUSION AND RECOMMENDATION

8.1 Conclusion

Although the Finite Analytic Method is still in its developmental stage, it has already demonstrated the advantage of invoking the local analytic solution of partial differential equation in constructing the numerical solution of linear or nonlinear partial differential equations. Chen and Li [1] and Chen and Naseri and Li [2] who initiated the development of the Finite Analytic method, have reported a great success in using this method for solving Navier-Stokes equations with the vorticity-stream function formulation. The Finite Analytic solution for Laplace equation [1], non-linear ordinary differential equation [10] and Poisson equation [11] have also been investigated. From these investigations it is seen that the FA method is accurate and has smaller numerical diffusion than the other numerical methods. Further, it converges well and is stable.

It was, therefore, with this belief that an FA solution of Navier-Stokes equations in primitive variables was considered in this study. Since the comparison of the FA method with other methods has been discussed in details by Cnen and Naseri and Li [2], no attempt was made to compare the results of the FA method with other methods. In this study, a new numerical procedure called the momentum averaging scheme was developed to accelerate the convergence of numerical solution for the

Navier Stokes equations formulated in u-v-p variables.

In this study, the Navier-Stokes equations are considered as a whole in a local element. The only approximation made in the equations is the local linearization. On each side of the element, the boundary condition is approximately represented by a second degree polynomial. From the first problem in Chapter 6, which considers the plate wake interaction, it was found that at moderate values of Reynolds numbers, i.e., 100, 400, and 800, the Navier-Stokes equations should be considered in order to simulate the interaction between the wake flow and boundary layer over the plate and to obtain correct solution from near wake to far wake regions.

In the second problem, the FA solution of the Navier-Stokes equations and the developed momentum averaging scheme are vigorously tested. They are used to solve for the closed square cavity flow where recirculation, separation and steep velocity gradients all exist. The numerical solution predicted by the present FA method compares favorably with the existing results.

From the present investigation it may be concluded that the FA numerical solution of the Navier-Stokes equations with the proposed momentum averaging scheme is accurate, stable and converges well. The method developed here for the two dimensional Navier-Stokes equations can be easily extended to the three dimensional case. It is, therefore, hoped that this study will pave the way for future investigations of the FA solution for three dimensional Navier-Stokes equations.

8.2 Recommendation

During this study, a number of difficulties arose in obtaining solutions and some critical decisions were made in order to get the final solutions. In this process a number of ideas were uncovered which are likely to improve the FA method. For example, as mentioned earlier, the grid size used in this method for solving Navier-Stokes equations depends on the Reynolds numbers. In order to bring the effect of the boundary layer in the flow field, at least one node was needed inside the boundary layer. This required that the grid size be very fine thereby increasing the computational time by a large amount. In order to reduce the computational time, the FA method needs to be developed to take nonuniform grid size so that in the boundary layer, the grid can be made as fine as required, and still maintaining a coarse grid outside the boundary layer.

Another way in which the FA solution technique can be improved is by obtaining an analytic solution which has simpler series solution than that obtained in Chapter 3. In Chapter 4, it was mentioned that before calculating the velocity from the momentum equation, the coefficients corresponding to the values of A and B are calculated from a subroutine for each element. Since the calculation of the coefficients involve series summation, a lot of computational time is needed by the FA method. To reduce this computational time, a new technique of obtaining the analytic solutions could be conceived. For example, the approximate function for the boundary condition need not be a second degree polynomial. Instead, it could be a linear combination of functions that satisfy the governing equation. This could lead to lesser computational time and better solutions.

APPENDIX A

COMPLETE FA SOLUTION OF POISSON EQUATION

In chapter III, the solutions to Poisson equation, momentum equation and continuity equation were simply written down. The solution to Poisson equations is obtained here in this Appendix. The other two equations are solved in Appendices B and C.

Before solving the equation here, an outline of the solution procedure is discussed briefly. The problem is divided into three simpler problems. Each of these problems is solved separately and the solutions are then added to obtain a solution of the Poisson equation.

The equation under consideration is

$$\nabla^2 p = 2(u_x v_y - v_x u_y)$$

with all four nonhomogeneous boundary conditions. The three simpler problems are:

(i) $\nabla^2 p_{1a} = 0$ with homogeneous boundary conditions at $y = \pm k$.

(ii) $\nabla^2 p_{1b} = 0$ with homogeneous boundary conditions at $x = \pm h$.

(iii) $\nabla^2 p_2 = 2(u_x v_y - v_x u_y)$ with homogeneous boundary conditions at $x = \pm h$ and $y = \pm k$.

Having solved these three problems, the solution to the Poisson equation is written as

$$p = p_{1a} + p_{1b} + p_2 \, .$$

The equation to be solved is

$$\nabla^2 p_{1a} = 0 \, , \tag{3.23}$$

and the boundary conditions are

$$p_{1a} = p_E(y) \text{ at } x = h \, ,$$

$$p_{1a} = p_W(y) \text{ at } x = -h,$$

$$p_{1a} = 0 \text{ at } y = \pm k \, .$$

The above equation is solved using separation of variables. The variable p_{1a} is assumed to be a product of two functions, i.e.,

$$p_{1a}(x,y) = X(x) \; Y(y) . \tag{A.1}$$

This is substituted in equation (3.23) and the resulting equation divided by $p_{1a}(x,y)$. This gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \, , \tag{A.2}$$

or

$$\frac{X''}{X} = - \frac{Y''}{Y} = \pm \mu^2 . \tag{A.3}$$

Since the boundary conditions on the north and south side are zero, the + sign is taken in equation (A.3). This leads to two equations

$$X'' - \mu^2 X = 0 \quad , \tag{A.4}$$

and

$$Y'' + \mu^2 Y = 0 \quad . \tag{A.5}$$

Equation (A.4) has a solution of the form

$$X(x) = C_1 \text{Sinh}\mu x + C_2 \text{Cosh}\,\mu x \quad , \tag{A.6}$$

and equation (A.5) has a solution of the form

$$Y(y) = D_1 \text{Sin}\mu y + D_2 \text{Cos}\mu y \; . \tag{A.7}$$

Therefore,

$$p_{1a}(x,y) = (C_1 \text{Sinh}\mu x + C_2 \text{Cosh}\mu x)(D_1 \text{Sin}\mu y + D_2 \text{Cos}\mu y). \tag{A.8}$$

The boundary condition at $y = -k$ is now substituted in (A.8)

$$p_{1a} = 0 = (C_1 \text{ Sinh}\mu x + C_2 \text{ Cosh}\mu x)(-D_1 \text{ Sin } \mu k + D_2 \text{ Cos } \mu k) ,$$

$$D_1 = D_2 \frac{\text{Cos } \mu k}{\text{Sin } \mu k}$$

Replacing D_1 in (A.8), p_{1a} (x,y) becomes

$$p_{1a}(x,y) = (C_1 \text{ Sinh}\mu x + C_2 \text{ Cosh}\mu x) \frac{D_2 \text{Sin}\mu(y+k)}{\text{Sin } \mu k} ,$$

or

$$p_{1a}(x,y) = (C_1 \text{ Sinh}\mu x + C_2 \text{ Cosh}\mu x) \frac{\text{Sin}\mu(y+k)}{\text{Sin } \mu k} , \tag{A.9}$$

where the constant D_2 is absorbed in C_1 and C_2. Now, at $y = k$, $p_{1a} = 0$. Therefore

$$p_{1a}(x,y) = 0 = (C_1 \text{ Sinh}\mu x + C_2 \text{ Cosh}\mu x) \frac{\text{Sin}\mu(2k)}{\text{Sin } \mu k} ,$$

$$\text{Sin}(2\mu k) = 0 ,$$

$$2\mu_n k = n\pi, \qquad n = 1,2,$$

and

$$\mu_n = \frac{n\pi}{2k} .$$

Therefore, $p_{1a}(x,y) = \sum_{n=1}^{\infty} [C_1 \text{ Sinh}\mu_n x + C_2 \text{ Cosh}\mu_n x] \dfrac{\text{Sin}\mu_n(y+k)}{\text{Sin}\mu_n k}$. (A.10)

The boundary condition at x = h is now substituted in equation (A.10). This gives

$$p_{1a}(x,y) = p_E(y) = \sum_{n=1}^{\infty} [C_{1n}\text{Sinh}\,\mu_n h + C_{2n}\text{Cosh}\,\mu_n h] * \frac{\text{Sin}\,\mu_n(y+k)}{\text{Sin}\,\mu_n k}. \tag{A.11}$$

But $p_E(y) = a_E + b_E y + c_E y^2$.

Multiplying both sides of equation (A.11) by Sin $\mu_m(y+k)$ and integrating with respect to y, the following relation is obtained

$$a_E \int_{-k}^{k} \text{Sin}\,\mu_m(y+k)dy + b_E \int_{-k}^{k} y\,\text{Sin}\,\mu_m(y+k)dy + c_E \int_{-k}^{k} y^2\,\text{Sin}\,\mu_m(y+k)dy =$$

$$\sum_{n=1}^{\infty} [C_{1n}\text{Sinh}\,\mu_n h + C_{2n}\text{Cosh}\,\mu_n h] \int_{-k}^{k} \text{Sin}\,\mu_n(y+k) * \text{Sin}\,\mu_m(y+k)dy. \tag{A.12}$$

Noting the orthogonal relation on the right side the above equation reduces to

$$a_E \int_{-k}^{k} \text{Sin}\,\mu_n(y+k)dy + b_E \int_{-k}^{k} y\,\text{Sin}\,\mu_n(y+k)dy + c_E \int_{-k}^{k} y^2\text{Sin}\,\mu_n(y+k)dy = [C_{1n}\text{Sinh}\,\mu_n h + C_{2n}\text{Cosh}\,\mu_n h]\,\frac{k}{\text{Sin}\mu_n k} \tag{A.13}$$

This is written as

$$\frac{\sin \mu_n k}{k} [a_E E_1 + b_E E_2 + c_E E_3] = C_{1n} \sinh \mu_n h + C_{2n} \cosh \mu_n h \,, \qquad (A.14)$$

where

$$E_1 = \int_{-k}^{k} \sin \mu_n (y+k) dy = \frac{1}{\mu_n} [1 - (-1)^n] \,,$$

$$E_2 = \int_{-k}^{k} \sin \mu_n (y+k) dy = - \frac{k}{\mu_n} [1 + (-1)^n] \,,$$

$$E_3 = \int_{-k}^{k} y^2 \sin \mu_n (y+k) dy = \frac{2}{\mu_n} (2k^2 - \frac{1}{\mu_n^2}) [1 - (-1)^n].$$

Similarly, using the boundary condition at $x = - h$, a relation is obtained which is

$$\frac{\sin \mu_n k}{k} [a_W E_1 + b_W E_2 + c_W E_3] = - C_{1n} \sinh \mu_n h + C_{2n} \cosh \mu_n h \,. \qquad (A.15)$$

C_{1n} and C_{2n} are now obtained from equations (A.14) and (A.15) so

$$C_{1n} = \frac{\sin \mu_n k}{2k \cosh \mu_n h} [a_E - a_W) E_1 + (b_E - b_W) E_2 + (c_E - c_W) E_3] \,,$$

and

$$C_{2n} = \frac{\sin \mu_n k}{2k \cosh \mu_n h} [(a_E + a_W) E_1 + (b_E + b_W) E_2 + (c_E + c_W) E_3].$$

Therefore,

$$p_{1a}(x,y) = \sum_{n=1}^{\infty} [C_{1n}\text{Sinh}\,\mu_n x + C_{2n}\text{Cosh}\,\mu_n x)\,\text{Sin}\,\mu_n(y+k), \tag{A.16}$$

where

$$C_{1n} = \frac{1}{2k\,\text{Sinh}\,\mu_n h}\,[(a_E - a_W)E_1 + (b_E - b_W)E_2 + (c_E - c_W)E_3], \tag{A.17}$$

and

$$C_{2n} = \frac{1}{2k\,\text{Cosh}\,\mu_n h}\,[(a_E + a_W)E_1 + (b_E + b_W)E_2 + (c_E + c_W)E_3]. \tag{A.18}$$

<u>Solution to Equation (3.24)</u>

The equation to be solved is

$$\nabla^2 p_{1b} = 0 \,, \tag{3.24}$$

and the boundary conditions are

$$p_{1b} = 0 \text{ at } x = \pm h \,,$$

$$p_{1b} = p_N(x) \text{ at } y = k \,,$$

$$p_{1b} = p_S(x) \text{ at } y = -k \,.$$

The solution to this equation is exactly similar to (A.16). So the solution can be written by simply replacing x, y, h, k and μ_n by y, x, k, h and ν_m. Therefore,

$$p_{1b}(x,y) = \sum_{m=1}^{\infty} [C_{3m} \text{Sinh } \nu_m y + C_{4m} \text{Cosh } \nu_m y] \text{ Sin } \nu_m (x + h), \qquad \text{(A.19)}$$

where

$$C_{3m} = \frac{1}{2h \text{ Sinh } \nu_m k} [(a_N - a_S) F_1 + (b_N - b_S) F_2 + (c_N - c_S) F_3] , \qquad \text{(A.20)}$$

and

$$C_{4m} = \frac{1}{2h \text{ Cosh } \nu_m k} [(a_N + a_S) F_1 + (b_N + b_S) F_2 + (c_N + c_S) F_3]. \qquad \text{(A.21)}$$

F_1, F_2 and F_3 in equations (A.20) and (A.21) are given by

$$F_1 = \int_{-h}^{h} \text{Sin } \nu_m (x+h) dx = \frac{1}{\nu_m} [1 - (-1)^m] ,$$

$$F_2 = \int_{-h}^{h} x \text{ Sin } \nu_m (x+h) dx = - \frac{h}{\nu_m} [1 + (-1)^m],$$

$$F_3 = \int_{-h}^{h} x^2 \text{ Sin } \nu_m (x+h) dx = \frac{2}{\nu_m} (2h^2 - \frac{1}{\nu_m^2}) [1 = (-1^m] ,$$

where

$$\nu_m = \frac{m\pi}{2h}, \quad m = 1,2,\ldots .$$

Solution to Equation (3.21)

The equation to be solved is

$$\nabla^2 p_2 = 2(u_x v_y - v_x u_y) , \tag{3.21}$$

and the boundary conditions are

$$p_2 = 0 \text{ at } x = \pm h ,$$

$$p_2 = 0 \text{ at } y = \pm k .$$

This equation is solved by expressing the nonhomogeneous term as a second degree polynomial. The term $2(u_x v_y - v_x u_y)$ is simply assumed to be a function of x and y. So

$$\nabla^2 p_2 = f(x,y) , \tag{A.22}$$

where

$$f(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^2 y^2 + a_7 xy^2 + a_8 x^2 y . \tag{A.23}$$

These 9 coefficients can be written in terms of the nodal values of f(x,y). As an example

$$f(h,0) = f_{EC} = a_0 + a_1 h + a_4 h^2 ,$$

$$f(0,0) = f_P = a_0 ,$$

$$f(-h,0) = f_{WC} = a_0 - a_1 h + a_4 h^2 ,$$

From these three equations, a_o, a_1 and a_4 are obtained. In this way, all the nine coefficients can be expressed in terms of the nodal values of $f(x,y)$. So

$$a_0 = f_P$$

$$a_1 = \frac{1}{2h} (f_{EC} - f_{WC})$$

$$a_2 = \frac{1}{2k} (f_{NC} - f_{SC})$$

$$a_3 = \frac{1}{4hk} (f_{NE} - f_{NW} - f_{SE} + f_{SW})$$

$$a_4 = \frac{1}{2h^2} (f_{EC} - 2f_p + f_{WC})$$

$$a_5 = \frac{1}{2k^2} (f_{NC} - 2f_p + f_{SC})$$

$$a_6 = \frac{1}{4h^2k^2} (f_{NE} + f_{SE} + f_{NW} + f_{SW} - 2f_{EC} - 2f_{WC} - 2f_{NC} - 2f_{SC} + 4f_p)$$

$$a_7 = \frac{1}{4hk^2} (f_{NE} + f_{SE} - f_{NW} - f_{SW} - 2f_{EC} + 2f_{WC})$$

$$a_8 = \frac{1}{4h^2k}(f_{NE} + f_{NW} - f_{SE} - f_{SW} + 2f_{SC} - 2f_{NC}).$$

The function f(x,y) is now represented by a Fourier series, i.e.,

$$f(x,y) = \sum_{n=1}^{\infty} D_n(y) \text{ Sin } \lambda_n(x + h) , \tag{A.24}$$

where $\lambda_n = \frac{n\pi}{2h}$, $n = 1,2,\ldots$.

Multiplying equation (A.24) by Sin $\lambda_m(\pi + h)$ and integrating with respect to x, the following relation is obtained.

$$\int_{-h}^{h} f(x,y) \text{ Sin } \lambda_m(x + h)\, dx = \sum_{n=1}^{\infty} D_n(y) \int_{-h}^{h} \text{Sin } \lambda_m(x + h) \text{ Sin } \lambda_n (x + h)dx. \tag{A.25}$$

Due to orthogonality of the sine function, the above equation reduces to

$$D_n(y)\, h = \int_{-h}^{h} f(x,y) \text{ Sin } \lambda_n(x+h)dx$$

$$= (a_0 + a_2 y + a_5 y^2) \int_{-h}^{h} \text{Sin } \lambda_n(n + h)dx$$

$$+ (a_1 + a_3 y + a_7 y^2) \int_{-h}^{h} x \text{ Sin } \lambda_n(x + h)dx$$

$$+ (a_4 + a_8 y + a_6 y^2) \int_{-h}^{h} x^2 \text{ Sin } \lambda_n(x + h)dx.$$

$$D_n(y)\, h = (a_0 + a_2 y + a_5 y^2)\, G_1$$

$$+ (a_1 + a_3 y + a_7 y^2)\, G_2$$

$$+ (a_4 + a_8 y + a_6 y^2)\, G_3 \quad ,$$

where

$$G_1 = \int_{-h}^{h} \mathrm{Sin}\ \lambda_n (x + h)\ dx = \frac{1}{\lambda_n} [1 - (-1)^n] \quad ,$$

$$G_2 = \int_{-h}^{h} x\ \mathrm{Sin}\ \lambda_n\ (x + h)\ dx = - \frac{h}{\lambda_n} [1 + (-1)^n] \ ,$$

$$G_3 = \int_{-h}^{h} x^2\ \mathrm{Sin}\ \lambda_n (x + h)\ dx = \frac{2}{\lambda_n} (2h^2 - \frac{1}{\mu_n^2}) [1 - (-1)^n].$$

So,

$$D_n(y) = \frac{1}{h} [(a_0 G_1 + a_1 G_2 + a_4 G_3) + (a_2 G_1 + a_3 G_2 + a_8 G_3) y$$

$$+ (a_5 G_1 + a_7 G_2 + a_6 G_3) y^2] \ , \qquad \text{(A.26)}$$

and

$$f(x,y) = \sum_{n=1}^{\infty} D_n(y)\ \mathrm{Sin}\ \nu_n (x + h) \ . \qquad \text{(A.24)}$$

It is now assumed that $p_2(x,y)$ is also of the form

$$p_2(x,y) = \sum_{n=1}^{\infty} B_n(y)\ \mathrm{Sin}\ \nu_n (x + h) \qquad \text{(A.27)}$$

Substituting equations (A.24) and (A.27) in the governing equation, i.e, equation (3.21), the following equation is obtained

$$\frac{\partial^2 B_n(y)}{\partial y^2} - \lambda_n^2 B_n(y) = D_n(y) \tag{A.28}$$

This is a non-homogeneous second order differential equation in $B_n(y)$. The solution consists of two parts, namely complementary solution and particular integral. Now,

$$D_n(y) = \hat{a}_0 + \hat{a}_1 y + \hat{a}_2 y^2 \tag{A.29}$$

where

$$\hat{a}_0 = \frac{1}{h}(a_0 G_1 + a_1 G_2 + a_4 G_3)$$

$$\hat{a}_1 = \frac{1}{h}(a_2 G_1 + a_3 G_2 + a_8 G_3)$$

$$\hat{a}_2 = \frac{1}{h}(a_5 G_1 + a_7 G_2 + a_6 G_3)$$

Since $D_n(y)$ is a second degree polynomial in y, the particular solution of $B_n(y)$ is also assumed to be a second degree polynomial in y. Thus

$$B_n(y)\Big|_p = C_7 + C_8 y + C_9 y^2 \tag{A.30}$$

Equations (A.29) and (A.30) are substituted in (A.28) to give

$$2C_9 - \lambda_n^2 (C_7 + C_8 y + C_9 y^2) = \hat{a}_0 + \hat{a}_1 y + \hat{a}_2 y^2 .$$

From this, C_7, C_8 and C_9 are evaluated. So

$$C_7 = -\frac{\hat{a}_0}{\lambda_n^2} - \frac{2\hat{a}_2}{\lambda_n^4} ,$$

$$C_8 = -\frac{\hat{a}_1}{\lambda_n^2} ,$$

$$C_9 = -\frac{\hat{a}_2}{\lambda_n^2} .$$

The complementary solution of $B_n(y)$ is

$$B_n(y) = A_1 e^{\lambda_n y} + B_1 e^{-\lambda_n y} .$$

Hence the complete solution to equation (A.28) is

$$B_n(y) = A_1 e^{\lambda_n y} + B_1 e^{-\lambda_n y} + C_7 + C_8 y + C_9 y^2 , \tag{A.31}$$

where

$$C_7 = \frac{1}{\lambda_n^2 h} (a_0 G_1 + a_1 G_2 + a_4 G_3) + \frac{2C_9}{\lambda_n^2} ,$$

$$C_8 = \frac{1}{\lambda_n^2 g} (a_2 G_1 + a_3 G_2 + a_8 G_3) , \tag{A.32}$$

$$C_9 = \frac{1}{\lambda_n^2 h} (a_5 G_1 + a_7 G_2 + a_6 G_3) .$$

The constants A_1 and B_1 are evaluated from the boundary conditions $B_n(y) = 0$ at $y = \pm k$. Therefore,

$$A_1 e^{\lambda_n k} + B_1 e^{-\lambda_n k} + C_7 + C_8 k + C_9 k^2 = 0 \quad ,$$

and

$$A_1 e^{-\lambda_n k} + B_1 e^{-\lambda_n k} + C_7 - C_8 k + C_9 k^2 = 0.$$

From these two equations A_1 and B_1 are found to be

$$A_1 = - C_7 \frac{\text{Sinh } \lambda_n k}{\text{Sinh } 2\lambda_n k} - C_8 k \frac{\text{Cosh } \lambda_n k}{\text{Sinh } 2\lambda_n k} - C_9 k^2 \frac{\text{Sinh } \lambda_n k}{\text{Sinh } 2\lambda_n k} ,$$

and

$$B_1 = - C_7 \frac{\text{Sinh } \lambda_n k}{\text{Sinh } 2\lambda_n k} + C_8 k \frac{\text{Cosh } \lambda_n k}{\text{Sinh } 2\lambda_n k} - C_9 k^2 \frac{\text{Sinh } \lambda_n k}{\text{Sinh } 2\lambda_n k} .$$

Finally, the solution for $p_2(x,y)$ is obtained as

$$p_2(x,y) = \sum_{n=1}^{\infty} [C_{5n} \text{Sinh } \lambda_n y + C_{6n} \text{Cosh } \lambda_n y + C_7 + C_8 y + C_9 y^2]$$

$$\text{Sin } \lambda_n (x + h) , \qquad \text{(A.33)}$$

where

$$C_{5n} = - \frac{C_8 k}{\text{Sinh } (\lambda k)} ,$$

and

$$C_{6n} = - \frac{C_7 + C_9 k^2}{\text{Cosh } (\lambda_n k)} .$$

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The three solutions obtained above are now summed to give the final solution to Poisson equation. Thus

$$p(x,y) = p_{1a} + p_{1b} + p_2 \ ,$$

and the value of p at the interior node p is

$$p_p = c_{NE}p_{NE} + c_{EC}p_{EC} + c_{SE}p_{SE} + c_{NC}p_{NC} + c_{SC}p_{SC}$$

$$+ c_{NW}p_{NW} + c_{WC}p_{WC} + c_{SW}p_{SW} + c_{NE}'f_{NE} + c_{EC}'f_{EC}$$

$$+ c_{SE}'f_{SE} + c_{NC}'f_{NC} + c_p'f_p + c_{SC}'f_{SC} + c_{NW}'f_{NW}$$

$$+ c_{WC}'f_{WC} + c_{SW}'f_{SW} \ .$$

The coefficients are given below.

$$c_{NE} = c_{SE} = c_{NW} = c_{SW} = \sum_{m=1,3}^{\infty} \frac{2}{m\pi} - \frac{16}{m^3\pi^3} \ S_m = 0.044685 \ ,$$

$$c_{EC} = c_{NC} = c_{WC} = c_{SC} = \sum_{m=1,3}^{\infty} \frac{16}{m^3\pi^3} \ S_m = 0.205315 \ ,$$

where

$$S_m = \frac{\mathrm{Sin}(m\pi/2)}{\mathrm{Cosh}(m\pi/2)} \ .$$

$c_{NE}' = c_{SE}' = c_{NW}' = c_{SW}'$

$$= h^2\left[\sum_{m=1,3}^{\infty} \left\{Q_m\left(\frac{1}{2} - \frac{32}{m^4\pi^4}\right) - \left(\frac{4}{m^2\pi^2} - \frac{32}{m^4\pi^4}\right)\right\}\frac{8\mathrm{Sin}(m\pi/2)}{m^3\pi}\right]$$

$$= 0.001895h^2 .$$

$$c_{NC}' = c_{SC}' = h^2\left[\sum_{m=1,3}^{\infty} \left\{Q_m\left(\frac{8}{m^2\pi^2} + \frac{64}{m^4\pi^4}\right) - \frac{64}{m^4\pi^4}\right\} \frac{8\mathrm{Sin}(m\pi/2)}{m^3\pi^3}\right]$$

$$= 0.01855h^2 .$$

$$c_{EC}' = c_{WC}' = h^2\left[\sum_{m,1,3}^{\infty} \left\{Q_m\left(\frac{64}{m^4\pi^4} - \frac{8}{m^2\pi^2}\right) - \left(\frac{64}{m^4\pi^4} - \frac{16}{m^2\pi^2}\right)\right\}\frac{8\mathrm{Sin}(m\pi/2)}{m^3\pi^3}\right]$$

$$= 0.01855h^2 .$$

$$c_p' = h^2\left[\sum_{m=1,3}^{\infty} \left\{Q_m\left(- \frac{128}{m^4\pi^4}\right) - \left(\frac{16}{m^2\pi^2} - \frac{128}{m^4\pi^4}\right)\right\}\frac{8\mathrm{Sin}(m\pi/2)}{m^4\pi^4}\right]$$

$$= 0.21289h^2 .$$

$Q_m = \dfrac{1}{\mathrm{Cosh}(m\pi/2)}$ and $h = k$.

APPENDIX B

COMPLETE FA SOLUTION OF MOMENTUM EQUATION

In section 3.3, the momentum equation was divided into three parts in order to obtain the analytical solution conveniently. In this Appendix, the solution to each of these three parts is obtained separately.

The discussion of section 3.3 is briefly reviewed here. The three problems to be solved are

Problem (1): Homogeneous equation (3.42) with two homogeneous boundary conditions.

Problem (2): Homogeneous equation (3.43) with other two homogeneous boundary conditions.

Problem (3): Non-homogeneous equation (3.44) with homogeneous boundary conditions.

The three solutions obtained from these three problems are then summed to give the final solution of the momentum equation.

Solution to equation (3.37)

The equation to be solved is

$$(A^2 + B^2)\bar{u}_1 = \bar{u}_{1xx} + \bar{u}_{1yy}, \tag{3.37}$$

along with the boundary conditions

$$\bar{u}_1(h,y) = (a_E + b_E y + c_E y^2)e^{-(Ah + By)} ,$$

$$\bar{u}_1(-h,y) = (a_W + b_W y + c_W y^2)e^{(Ah - By)} ,$$

$$\bar{u}_1(x,k) = 0 ,$$

$$\bar{u}_1(x,-k) = 0 .$$

Since the boundary conditions on the northern and southern sides are homogeneous, the solution can be assumed to be of the form

$$\bar{u}_1(x,y) = \sum_{n=1}^{n} A_n(x) \text{ Sin } \lambda_n(y + k) , \tag{B.1}$$

where

$$\lambda_n = \frac{n\pi}{2k}, \; n = 1,2,\ldots \quad ,$$

and A(x) is a function to be obtained. Equation (B.1) is now substituted in equation (3.27) to give

$$\sum_{n=1}^{\infty} A''_n(x)\text{Sin } \lambda_n(y + k) - \sum_{n=1}^{\infty}\lambda_n^2 A_n(x) \text{ Sin } \lambda_n(y + k)$$
$$- (A^2 + B^2) \sum_{n=1}^{\infty} A_n(x) \text{ Sin } \lambda_n(y + k) = 0 , \tag{B.2}$$

or

$$A''_n(x) - (A^2 + B^2 + \lambda_n{}^2)A_n(x) = 0 ,$$

or

$$A''_n(x) - q_n{}^2 A_n(x) = 0 , \tag{B.3}$$

where

$$q_n{}^2 = A^2 + B^2 + \lambda_n{}^2 \quad .$$

Equation (B.3) is a second order, homogeneous, ordinary differential equation which has a solution of the form

$$A_n(x) = C_n \mathrm{Sinh}(q_n x) + C_{2n} \mathrm{Cosh}(q_n x) . \tag{B.4}$$

The constants C_{1n} and C_{2n} are evaluated from the boundary conditions at $x = \pm h$. At $x = h$,

$$\bar{u}_1(h,y) = (a_E + b_E y + c_E y^2) e^{-(Ah + By)}$$

$$= \sum_{n=1}^{\infty} A_n(h) \mathrm{Sin}\, \lambda_n (y + k) . \tag{B.5}$$

Multiplying both sides by $\mathrm{Sin}\, \lambda_m(y + k)$ and integrating with respect to y results in

$$\int_{-k}^{k} (a_E + b_E y + c_E y^2) e^{-(Ah + By)} \mathrm{Sin}\, \lambda_m (y + k)$$

$$= \sum_{n=1}^{\infty} A_n(h) \int_{-k}^{k} \mathrm{Sin}\, \lambda_n (y + k) \mathrm{Sin}\, \lambda_m (y + k)\, dy \quad .$$

Due to orthogonality of the Sine function, the above equation reduces to

$$A_n(h) = \frac{1}{k} \int_{-k}^{k} (a_E + b_E y + c_E y^2) e^{-(Ah + By)} \mathrm{Sin}\, \lambda_n (y + k)\, dy ,$$

or

$$A_n(h) = \frac{e^{-Ah}}{k} [a_E \bar{E}_1 + b_E \bar{E}_2 + c_E \bar{E}_3] , \tag{B.6}$$

where

$$\bar{E}_1 = \int_{-h}^{h} e^{-By}\,\mathrm{Sin}\,\lambda_n(y + k)dy = \frac{e^{Bk}\,\lambda_n[1 - (-1)^n e^{-2Bk}}{(B^2 + \lambda_n^2)} \;,$$

$$\bar{E}_2 = \int_{-h}^{h} ye^{-By}\,\mathrm{Sin}\,\lambda_n(y + k)\,dy$$

$$= e^{Bk}\left\{\left[-\frac{n\pi(-1)^n e^{-2Bk}}{B^2 + \lambda_n^2}\right] + \left(\frac{2B}{B^2 + \lambda_n^2} - k\right)\left[\frac{1-(-1)^n e^{-2Bk}}{B^2 + \lambda_n^2}\right]\lambda_n\right\},$$

$$\bar{E}_3 = \int_{-h}^{h} y^2e^{-By}\,\mathrm{Sin}\,\lambda_n(y + k)dy$$

$$= e^{Bk}\left\{-\frac{n\pi k(-1)^n e^{-2Bk}}{(B^2 + \lambda_n^2)} - \frac{4Bn\pi(-1)^n e^{-2Bk}}{(B^2 + \lambda_n^2)^2}\right.$$

$$+ \frac{2kn\pi(-1)^n e^{-2Bk}}{(B^2 + \lambda_n^2)} + \left[\frac{2(3B^2 - \lambda_n^2)}{(B^2 + \lambda_n^2)} - \frac{4Bk}{(B^2 + \lambda_n^2)} + k^2\right]*$$

$$\left.\left[\frac{1-(-1)^n e^{-2Bk}}{(B^2 + \lambda_n^2)}\right]\lambda_n\right\} \;.$$

Similarly, at $x = -h$,

$$A_n(-h) = \frac{e^{Ah}}{k}\,(a_W\bar{E}_1 + b_W\bar{E}_2 + c_W\bar{E}_3)\;. \qquad \text{(B.7)}$$

Returning to equation (B.4), the constants C_{1n} and C_{2n} are now evaluated from $A_n(x)$ at $x = + h$. So

$$A_n(h) = C_{1n}\mathrm{Sinh}(q_nh) + C_{2n}\mathrm{Cosh}(q_nh) \;,$$

$$A_n(-h) = -C_{1n}\mathrm{Sinh}(q_nh) + C_{2n}\mathrm{Cosh}(q_nh)\,.$$

From these two equations

$$C_{1n} = \frac{A_n(h) - A_n(-h)}{2\mathrm{Sinh}(q_n h)} , \tag{B.8}$$

and

$$C_{2n} = \frac{A_n(h) + A_n(-h)}{2\mathrm{Sinh}(q_n h)} . \tag{B.9}$$

Therefore,

$$\bar{u}_1(x,y) = \sum_{n=1}^{\infty}\{C_{1n}\mathrm{Sinh}(q_n h) + C_{2n}\mathrm{Cosh}(q_n h)\}\mathrm{Sin}\ \lambda_n(y + k) , \tag{B.10}$$

where C_{1n} and C_{2n} are given by equations (B.8) and (B.9).

Solution to equation (3.38)

The equation to be solved is

$$(A^2 + B^2)\ \bar{u}_2 = \bar{u}_{2xx} + \bar{u}_{2yy} , \tag{3.38}$$

and the boundary conditions are

$$\bar{u}_2(h,y) = 0 ,$$

$$\bar{u}_2(-h,y) = 0 ,$$

$$\bar{u}_2(x,k) = (a_N + b_N x + c_N x^2)e^{-(Ax + Bk)} ,$$

$$\bar{u}_2(x,-k) = (a_S + b_S x + c_S x^2)e^{-(Ax - Bk)} ,$$

The solution to this equation is exactly similar to (B.10). So the solution can be written by simply replacing x, y, h, k, A, B, λ_n and q_n by y, x, k, h, B, A, μ_m and q_m. Therefore

$$\bar{u}_2(x,y) = \sum_{m=1}^{\infty} \{C_{1m}\mathrm{Sinh}(q_m y) + C_{2m}\mathrm{Cosh}(q_m y)\}\mathrm{Sin}\ \mu_m(x + h)\ , \tag{B.11}$$

where

$$C_{1m} = \frac{B_m(k) - B_m(-k)}{2\mathrm{Sinh}(q_m k)}\ , \tag{B.12}$$

$$C_{2m} = \frac{B_m(k) + B_m(-k)}{2\mathrm{Cosh}(q_m k)}\ . \tag{B.13}$$

$B_m(k)$ and $B_m(-k)$ in the above equations are given by

$$B_m(k) = \frac{e^{-Bk}}{h}(a_N\bar{F}_1 + b_N\bar{F}_2 + c_N\bar{F}_3)\ , \tag{B.14}$$

$$B_m(-k) = \frac{e^{Bk}}{h}(a_S\bar{F}_1 + b_S\bar{F}_2 + c_S\bar{F}_3)\ . \tag{B.15}$$

$\bar{F}_1$, $\bar{F}_2$ and $\bar{F}_3$ in equations (B.14) and (B.15), are, in turn, given by

$$\bar{F}_1 = \int_{-h}^{h} e^{-Ax}\ \mathrm{Sin}\ \mu_m(x + h)dx = \frac{e^{Ah}\mu_m[1 - (-1)^m e^{-2Ah}]}{(A^2 + \mu_m^2)}\ ,$$

$$\bar{F}_2 = \int_{-h}^{h} xe^{-Ax}\ \mathrm{Sin}\ \mu_m(x + h)dx$$

$$= e^{Bk}\{[-\frac{m\pi(-1)^m e^{-2Ah}}{(A^2 + \mu_m^2}] + (\frac{2A}{A^2 + \mu_m^2} - h)[\frac{1-(-1)^m e^{-2Ah}}{A^2 + \mu_m^2}]\ _m\}\ ,$$

$$\bar{F}_3 = \int_{-h}^{h} x^2 e^{-Ax}\mathrm{Sin}\ \mu_m(x + h)dx$$

$$= e^{Ah} \{ - \frac{m\pi h(-1)^m e^{-2Ah}}{(A^2 + \mu_m^2)} - \frac{4Am\pi(-1)^m e^{-2Ah}}{(A^2 + \mu_m^2)^2} + \frac{2hm\pi(-1)^m e^{-2Ah}}{(A^2 + \mu_m^2)}$$

$$+ [\frac{2(3A^2 - \mu_m^2)}{(A^2 + \mu_m^2)^2} - \frac{4Ah}{(A^2 + \mu_m^2)} + h^2][\frac{1-(-1)^m e^{-2Ah}}{(A^2 + \mu_m^2)}] \mu_m \} \quad ,$$

$$q_m^2 = A^2 + B^2 + \mu_m^2 \ ,$$

$$\mu_m = \frac{m\pi}{2h}; \quad m = 1,2,\dots \ .$$

Solution to equation (3.39)

The equation to be solved is

$$(A^2 + B^2)\bar{u}_3 = \bar{u}_{3xx} + \bar{u}_{3yy} - Re\ p_x\ e^{-(Ax + By)} \quad , \tag{3.39}$$

and the boundary conditions are

$$\bar{u}_3(h,y) = 0 \quad ,$$
$$\bar{u}_3(-h,y) = 0 \quad ,$$
$$\bar{u}_3(x,k) = 0 \quad ,$$
$$\bar{u}_3(x,-k) = 0 \quad .$$

Let $g(x,y) = Re\ p_x\ e^{-(Ax + By)} = \sum_{\ell=1}^{\infty} C_\ell(y) \text{ Sin } \nu_\ell(x + h)$

Multiplying both sides by Sin $\nu_m(x + h)$ and integrating gives

$$(Re\ p_x) \int_{-h}^{h} e^{-(Ax + By)} \text{ Sin } \nu_\ell(x + h)\ dx = C_\ell(y)(h)$$

$$C\ (y) = \frac{Re\ p_x}{h} e^{-By} \int_{-h}^{h} e^{-Ax} \operatorname{Sin} \nu_\ell (x + h) dx$$

$$= \frac{Re\ p_x}{h} I\ e^{-By} ,$$

where

$$I = \int_{-h}^{h} e^{-Ax} \operatorname{Sin} \nu_\ell (x + h) dx$$

$$= \frac{\nu_\ell e^{Ah} - \nu_\ell e^{-Ah} (-1)^\ell}{A^2 + \nu_\ell^2} .$$

Assuming $\bar{u}_3$ of the form

$$\bar{u}_3 = \sum_{\ell=1}^{\infty} Y_\ell(y) \operatorname{Sin} \nu_\ell (x + h) ,$$

$g(x,y)$ and $\bar{u}_3$ are substituted in equation (3.39). This gives

$$\Sigma (A^2 + B^2) Y_\ell(y) \operatorname{Sin} \nu_\ell (x + h) = \sum_{\ell=1}^{\infty} (-\nu_\ell^2) Y_\ell(y) \operatorname{Sin} \nu_\ell (x + h)$$

$$+ \sum_{\ell=1}^{\infty} Y_\ell''(y) \operatorname{Sin} \nu_\ell (x + h)$$

$$- \sum_{\ell=1}^{\infty} C_\ell(y) \operatorname{Sin} \nu_\ell (x + h) .$$

From this

$$Y_\ell''(y) - q_\ell^2 Y_\ell(y) = C_\ell(y) , \qquad \text{(B.16)}$$

where

$$q_\ell^2 = A^2 + B^2 + \nu_\ell^2 .$$

Equation (B.16) has a solution of the form

$$Y_\ell(y) = C_{3\ell}e^{q_\ell y} + C_{4\ell}e^{-q_\ell y} + C_{5\ell}\,e^{-By} \quad . \tag{B.17}$$

The first two terms on the right hand side represent the complementary solution and the third term is the particular solution. To obtain the constant $C_{5\ell}$, the term $C_{5\ell}e^{-By}$ is substituted in equation (B.16). This gives

$$C_{5\ell} = \frac{1}{(B^2 - q_\ell^{\,2})}\left\{\frac{Re\; p_x I}{h}\right\}.$$

To evaluate the constants $C_{3\ell}$ and $C_{4\ell}$, the two conditions used are $Y_\ell(y) = 0$ at $y = \pm\, k$. Thus

$$C_{3\ell}\,e^{q_\ell k} + C_{4\ell}e^{-q_\ell k} + C_{5\ell}e^{-Bk} = 0\ , \tag{B.18}$$

and

$$C_{3\ell}e^{-q_\ell k} + C_{4\ell}e^{q_\ell k} + C_{5\ell}e^{Bk} = 0\ . \tag{B.19}$$

Equations (B.18) and (B.19) are solved for $C_{3\ell}$ and $C_{4\ell}$. Therefore,

$$C_{3\ell} = C_{5\ell}\,\frac{\text{Sinh}(q_\ell - B)k}{\text{Sinh}\;2q_\ell k} \quad ,$$

and

$$C_{4\ell} = C_{5\ell}\,\frac{\text{Sinh}\;(q_\ell + B)k}{\text{Sinh}\;2q_\ell k} \quad ,$$

$$Y_\ell(y) = C_{5\ell}\left\{\frac{\text{Sinh}(q_\ell - B)ke^{q_\ell y} + \text{Sinh}(q_\ell + B)ke^{-q_\ell y}}{\text{Sinh}\;2q_\ell k} - e^{-By}\right\}. \tag{B.20}$$

$$\bar{u}_3(x,y) = \sum_{\ell=1}^{\infty} Y_\ell(y)\, \mathrm{Sin}\,\nu_\ell(x + h). \tag{B.21}$$

Final Solution

The complete solution to the momentum equation in an element is given by

$$u(x,y) = \{\bar{u}_1(x,y) + \bar{u}_2(x,y) + \bar{u}_3(x,y)\}e^{(Ax + By)}. \tag{B.22}$$

To evaluate the velocity at the interior node P, x = 0 and y = 0 are substituted in the above equation. This gives

$$u_P = \bar{u}_{1P} + \bar{u}_{2P} + \bar{u}_{3P}\ . \tag{B.23}$$

Evaluating equations (B.10), (B.11) and (B.21) at the node P and substituting in (B.23) gives after some rearrangement

$$\begin{aligned} u_P &= C_{NE}U_{NE} + C_{EC}U_{EC} + C_{SE}U_{SE} + C_{NC}U_{NC} + C_{SC}U_{SC} \\ &\quad + C_{NW}U_{NW} + C_{WC}U_{WC} + C_{SW}U_{SW} + C_P(\mathrm{Re}\ p_x)_P\ . \end{aligned} \tag{B.24}$$

This is the 9-point FA solution of the Navier-Stokes equation. The coefficients in equation (B.24) are given by

$$C_{NE} = \sum_{n=1,3,..}^{\infty} S_m\{\frac{e^{-Ah}}{2k^2}(E_2 + \frac{E_3}{k}) + \frac{e^{-Bk}}{2h^2}(\bar{E}_2 + \frac{\bar{E}_3}{h})\}\mathrm{Sin}(n\pi/2)$$

$$C_{EC} = \sum_{n=1,3..}^{\infty} S_m\{\frac{e^{-Ah}}{k}(E_1 - \frac{E_3}{k^2})\}\mathrm{Sin}(n\pi/2)$$

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$$C_{SE} = \sum_{n=1,3}^{\infty} S_m \{\frac{e^{-Ah}}{2k^2} (-E_2 + \frac{E_3}{k}) + \frac{e^{Bk}}{2h^2} (\bar{E}_2 + \frac{\bar{E}_3}{h})\} \text{ Sin } (n\pi/2)$$

$$C_{NC} = \sum_{n=1,3}^{\infty} S_m \{\frac{e^{-Bk}}{h} (\bar{E}_1 - \frac{\bar{E}_3}{h^2})\} \text{Sin } (n\pi/2)$$

$$C_{SC} = \sum_{n=1,3}^{\infty} S_m\{\frac{e^{Bk}}{h}(\bar{E}_1 - \frac{\bar{E}_3}{h^2})\} \text{Sin}(n\pi/2)$$

$$C_{NW} = \sum_{n=1,3}^{\infty} S_m\{\frac{e^{Ah}}{2k^2} (E_2 - \frac{E_3}{k}) + \frac{e^{-Bk}}{2h^2}(-\bar{E}_2 + \frac{\bar{E}_3}{h})\} \text{Sin}(n\pi/2)$$

$$C_{WC} = \sum_{n=1,3}^{\infty} S_m\{\frac{e^{Ah}}{k} (E_1 - \frac{E_3}{k^2})\} \text{Sin}(n\pi/2)$$

$$C_{SW} = \sum_{n=1,3}^{\infty} S_m\{\frac{e^{Ah}}{2k^2} (-E_2 + \frac{E_3}{k}) + \frac{e^{Bk}}{2h^2} (-\bar{E}_2 + \frac{\bar{E}_3}{k})\} \text{Sin}(n\pi/2)$$

$$C_p = \sum_{\ell=1}^{\infty} \frac{\lambda_\ell(e^{Ah} - (-1)^\ell e^{-Ah})}{h(A^2 + \lambda_\ell^2)^2} [\frac{\text{Sinh}(q_\ell - B)k + \text{Sinh}(q_\ell + B)k}{\text{Sinh } 2q_\ell k} - 1] \text{ Sin}(\ell\pi/2)$$

$$S_m = \frac{\text{Sin }_m h}{\text{Cosh } q_\ell k} .$$

APPENDIX C

SOLUTION OF CONTINUITY EQUATION

In this Appendix, the solution for the continuity equation is obtained. As mentioned in section 3.4, the analytic solution of u(x,y) in an element is not used to calculate v from the continuity equation. Instead, u(x,y) is approximated by a polynomial and substituted in the continuity equation which is then integrated to give the solution for v for an element. Therefore,

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2$$
$$+ a_6 x^2 y^2 + a_7 xy^2 + a_8 x^2 y . \qquad \text{(C.1)}$$

The coefficients in the above equation are written in terms of the surrounding nodal values, i.e.,

$$a_0 = u_P$$

$$a_1 = \frac{1}{2h} (u_{EC} - u_{WC})$$

$$a_2 = \frac{1}{2k} (u_{NC} - u_{SC})$$

$$a_3 = \frac{1}{4hk} (u_{NE} - u_{NW} - u_{SE} + u_{SW})$$

$$a_4 = \frac{1}{2h^2} (u_{EC} - 2u_P + u_{WC})$$

$$a_5 = \frac{1}{2k^2}(u_{NC} - 2u_P + u_{SC})$$

$$a_6 = \frac{1}{4h^2k^2}(u_{NE} + u_{SE} + u_{NW} + u_{SW} - 2u_{EC} - 2u_{WC} - 2u_{NC} - 2u_{SC} + 4u_P)$$

$$a_7 = \frac{1}{4hk^2}(u_{NE} + u_{SE} - u_{NW} - u_{SW} - 2u_{EC} + 2u_{WC})$$

$$a_8 = \frac{1}{4h^2k}(u_{NE} + u_{NW} - 2u_{NC} - u_{SE} - u_{SW} + 2u_{SC}) \ .$$

Differentiating equation (C.1) with respect to x gives

$$u_x = a_1 + a_3y + 2a_4x + 2a_6xy^2 + a_7y^2 + 2a_8xy \ . \tag{C.3}$$

The value of u_x is now evaluated at each of the nine elements. So

$$\begin{aligned}
(u_x)_P &= u_x(0,0) = a_1 \\
(u_x)_{EC} &= u_x(h,0) = a_1 + 2a_4h \\
(u_x)_{WC} &= u_x(-h,0) = a_1 - 2a_4h \\
(u_x)_{NC} &= u_x(0,k) = a_1 + a_3k + a_7k^2 \\
(u_x)_{SC} &= u_x(0,-k) = a_1 - a_3k + a_7k^2 \\
(u_x)_{NE} &= u_x(h,k) = a_1 + a_3k + 2a_4h + 2a_6hk^2 \\
&\qquad + a_7k^2 + 2a_8hk \\
(u_x)_{NW} &= u_x(-h,k) = a_1 + a_3k = 2a_4h - 2a_6hk^2 \\
&\qquad + a_7k^2 - 2a_8hk
\end{aligned} \tag{C.4}$$

$$(u_x)_{SE} = u_x(h,-k) = a_1 - a_3k + 2a_4h + 2a_6hk^2 + a_7k^2 - 2a_8hk$$

$$(u_x)_{SW} = u_x(-h,-k) = a_1 - a_3k - 2a_4h - 2a_6hk^2 + a_7k^2 + 2a_8hk \, .$$

Substituting equation (C.2) in (C.4) gives

$$\begin{aligned}
(u_x)_P &= \frac{1}{2h}(u_{EC} - u_{WC}) \\
(u_x)_{EC} &= \frac{1}{2h}(3u_{EC} - 4u_P + u_{WC}) \\
(u_x)_{WC} &= \frac{1}{2h}(-u_{EC} + 4u_P - 3u_{WC}) \\
(u_x)_{NC} &= \frac{1}{2h}(u_{NE} - u_{NW}) \\
(u_x)_{SC} &= \frac{1}{2h}(u_{SE} - u_{SW}) \\
(u_x)_{NE} &= \frac{1}{2h}(3u_{NE} + u_{NW} - 4u_{NC}) \\
(u_x)_{NW} &= \frac{1}{2h}(-u_{NE} = 3u_{NW} + 4u_{NC}) \\
(u_x)_{SE} &= \frac{1}{2h}(3u_{SE} + u_{SW} - 4u_{SC}) \\
(u_x)_{SW} &= \frac{1}{2h}(-u_{SE} - 3u_{SW} + 4u_{SC}) \, .
\end{aligned} \tag{C.5}$$

Now, a second degree polynomial is written for the derivative u_x, i.e.,

$$u_x = \bar{a}_0 + \bar{a}_1x + \bar{a}_2y + \bar{a}_3xy + \bar{a}_4x^2 + \bar{a}_5y^2 + \bar{a}_6x^2y + \bar{a}_7xy^2 + a_8x^2y. \tag{C.6}$$

The coefficients in equation (C.6) are expressed in terms of the derivatives at the nodes as given by equation (C.5). So

$$\bar{a}_0 = (u_x)_P$$

$$\bar{a}_1 = \frac{1}{2h}\{(u_x)_{EC} - (u_x)_{WC}\}$$

$$\bar{a}_2 = \frac{1}{2k}\{(u_x)_{NC} - (u_x)_{SC}\}$$

$$\bar{a}_3 = \frac{1}{4hk}\{(u_x)_{NE} - (u_x)_{NW} - (u_x)_{SE} + (u_x)_{SW}\}$$

$$\bar{a}_4 = \frac{1}{2h^2}\{(u_x)_{EC} - 2(u_x)_P + (u_x)_{WC}\}$$

$$\bar{a}_5 = \frac{1}{2k^2}\{(u_x)_{NC} - 2(u_x)_P + (u_x)_{SC}\}$$

$$\bar{a}_6 = \frac{1}{4h^2k^2}\{(u_x)_{NE} + (u_x)_{SE} + (u_x)_{NW} + (u_x)_{SW} - 2(u_x)_{EC} - 2(u_x)_{WC} - 2(u_x)_{NC} - 2(u_x)_{SC} + 4(u_x)_P\}$$

$$\bar{a}_7 = \frac{1}{4hk^2}\{(u_x)_{NE} + (u_x)_{SE} - (u_x)_{NW} - (u_x)_{SW} - 2(u_x)_{EC} + 2(u_x)_{WC}\}$$

$$\bar{a}_8 = \frac{1}{4h^2k}\{(u_x)_{NE} + (u_x)_{NW} - 2(u_x)_{NC} - (u_x)_{SE} - (u_x)_{SW} + 2(u_x)_{SC}\} .$$

Integrating the continuity equation gives

$$v\Big|_{SC}^{P} = -\int_{-k}^{0} u_x(0,y)\,dy \quad ,$$

$$v_P - v_{SC} = -\int_{-k}^{0} [a_0 + a_2y + a_5y^2]\,dy$$

$$= - [\bar{a}_0 y + \frac{\bar{a}_2}{2} y^2 + \frac{\bar{a}_5}{3} y^3]_{-k}^{0}$$

$$= - (u_x)_P\, k + \frac{1}{4} k \{(u_x)_{NC} - (u_x)_{SC}\} - \frac{1}{6} k\{(u_x)_{NC} - 2(u_x)_P + (u_x)_{SC}\}$$

$$= \frac{k}{12} [(u_x)_{NC} - 8(u_x)_P - 5(u_x)_{SC}]$$

$$= \frac{k}{12h} [\frac{1}{2} u_{NE} - \frac{1}{2} u_{NW} - \frac{4}{3} u_{EC} + \frac{4}{3} u_{WC} - \frac{5}{2} u_{SE} + \frac{5}{2} u_{SW}] \ .$$

If $h = k$,

$$v_P - v_{SC} = \frac{1}{24} u_{NE} - \frac{1}{24} u_{NW} = \frac{5}{24} u_{SE} - \frac{5}{24} u_{SW} - \frac{1}{3} u_{EC} + \frac{1}{3} u_{WC}. \quad (C.8)$$

Similarly, integrating thecontinuity equation from the node NC to the node P gives

$$v_P - u_{NC} = \frac{5}{24} u_{NE} - \frac{5}{24} u_{NW} - \frac{1}{24} u_{SE} + \frac{1}{24} u_{SW} + \frac{1}{3} u_{EC} - \frac{1}{3} u_{WC}. \quad (C.9)$$

Combining equations (C.8) and (C.9) gives

$$v_P = \frac{1}{2} (v_{NC} + v_{SC}) + \frac{1}{8} (u_{NE} - u_{NW} - u_{SE} + u_{SW}). \qquad (C.10)$$

Equation (C.10) is used for calculating the velocity v_P if the velocity u_P has been calculated using x-momentum equation. If, on the other hand, v_P is calculated from the y-momentum equation, u_P is obtained from the continuity equation. The equation used, which is identical to equation (C.10) is

$$u_P = \frac{1}{2}(u_{WC} + u_{EC}) + \frac{1}{8}(v_{NE} - v_{SE} - v_{NW} + v_{SW}) .$$

APPENDIX D

COMPUTER PROGRAM

```
C*******************************************************
C       THIS PROGRAM SOLVES THE NAVIER-STOKES
C       EQUATIONS FOR A TWO DIMENSIONAL CAVITY
C       USING PRIMITIVE VARIABLES(U,V,P). THE
C       FINITE ANALYTIC METHOD IS USED TO
C       OBTAIN LINEAR ALGEBRIAC EQUATIONS
C       FROM THE GOVERNING EQUATIONS WHICH ARE
C       THEN SOLVED BY THE IMPLICIT AND GAUSS-
C       SIEDEL NUMERICAL METHODS. THIS PROGRAM
C       WAS RUN ON PRIME-750 COMPUTER.
C*******************************************************
C           LIST OF VARIABLES USED IN THE PROGRAM
C
C
C       RE      REYNOLDS NO. BASED ON SOME REFERENCE
C               LENGTH AND VELOCITY SCALE
C       PRESSO  OLD VALUE OF PRESSURE
C       PRESSN  NEW VALUE OF PRESSURE
C       UOLD    OLD VALUE OF U-VELOCITY
C       UNEW    NEW VALUE OF U-VELOCITY
C       VOLD    OLD VALUE OF V-VELOCITY
C       VNEW    NEW VALUE OF V-VELOCITY
C       IZMAX   MAX. NO. OF NODES IN X-DIRECTION
C       IRMAX   MAX. NO. OF NODES IN Y-DIRECTION
C       ITERP   MAX. NO. OF ITERATIONS FOR PRESSURE
C       ITRV    MAX. NO. OF ITERATIONS FOR VELOCITIES
C       IEND    MAX. NO. OF OVERAL ITERATIONS
C       UP      VELOCITY OF MOVING PLATE
C       TOLUV   CONVERGENCE CRITERION FOR VELOCITIES
C       RFP     RELAXATION PARAMETER FOR PRESSURE
C       RFU     RELAXATION PARAMETER FOR U-VELOCITY
C       RFV     RELAXATION PARAMETER FOR V-VELOCITY
C
C*******************************************************
$INSERT SYSCOM>ERRD.F
$INSERT SYSCOM>KEYS.F
$INSERT SYSCOM>A$KEYS
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      COMMON/COMA/ PRESSN(61,61),UNEW(61,61),VNEW(61,61),
     $FNEW(61,61),BBB(61),AAA(61),CC(61),C(61),D(61),
     $T(61),AA(61)
      COMMON/COMB/ ZETA(61,61),PSIN(61,61),PS(61),
     $PRESSO(61,61),UOLD(61,61),VOLD(61,61)
      COMMON/COMC/ CP1P1,CP1P0,CP1M1,CM1P1,
     $CM1P0,CM1M1,CP0P1,CP0M1
      COMMON/COMD/DCM1P1(74,74),DCM1P0(74,74),
```

```
     $DCM1M1(74,74),DCP0P1(74,74),DCP0M1(74,74),
     $DCP1P1(74,74),DCP1P0(74,74),DCP1M1(74,74)
      COMMON/COME/COEFFP(61,61)
      COMMON/COMF/A,B,IZZ,IRR,AKK,BKK
     $,ITER,IZMAX,IRMAX,DX,DY
      DATA CNE/0.044685/,CNW/0.044685/,CSE/0.044685/,
     $CSW/0.044685/,CEC/0.205315/,CWC/0.205315/,
     $CNC/0.205315/,CSC/0.205315/,
     $FNE/0.001895076/,FNW/0.001895076/,FSE/
     $0.001895076/,FSW/0.001895076/,FEC/0.01855256/,
     $FWC/0.01855256/,FNC/0.01855256/,FSC/0.01855256/,
     $FP/0.2128948/
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C     INPUT STATEMENTS
      CALL SRCH$$(K$READ,'INPT',4,7,TYPE,CODE)
      CALL SRCH$$(K$WRIT,'OTPT',4,2,TYPE,CODE)
      READ(11,*)IZMAX,IRMAX,ITERP,IEND,ITRV
      READ(11,*)X,Y,UP,RE,TOLUV
      DO 100 IZ=1,74
      DO 100 IR=1,74
      READ(11,500)DCM1P1(IZ,IR),DCM1P0(IZ,IR),DCM1M1(IZ,IR),
     $          DCP0P1(IZ,IR),DCP0M1(IZ,IR),
     $          DCP1P1(IZ,IR),DCP1P0(IZ,IR),DCP1M1(IZ,IR)
 100  CONTINUE
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      WRITE(6,501) IZMAX,IRMAX,ITERP,IEND,ITRV,
     $X,Y,UP,RE,TOLUV
C
C
C
      RFP=1.0
      RFU=1.0
      RFV=1.0
      DX=1.0/X
      DY=1.0/Y
      WRITE(6,502) DX,DY
      IZM1=IZMAX-1
      IRM1=IRMAX-1
      IRM2=IRM1-1
      IZM2=IZM1-1
      IZMM=(IZMAX+1)/2
      IZMM1=IZMM-1
      IZMP1=IZMM+1
      M=IZM1
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C     INITIAL GUESS FOR VELOCITY AND PRESSURE
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 105 IR=2,IRM1
      DO 105 IZ=2,IZM1
      UNEW(IZ,IR)=0.0
      VNEW(IZ,IR)=0.0
```

```
      PRESSN(IZ,IR)=0.0
 105  CONTINUE
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C     BOUNDARY CONDITIONS FOR U,V FOR CAVITY
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 110 IZ=1,IZMAX
      UNEW(IZ,1)=UP
      VNEW(IZ,1)=0.0
      UNEW(IZ,IRMAX)=0.0
      VNEW(IZ,IRMAX)=0.0
 110  CONTINUE
      DO 115 IR=2,IRM1
      UNEW(1,IR)=0.0
      VNEW(1,IR)=0.0
      UNEW(IZMAX,IR)=0.0
      VNEW(IZMAX,IR)=0.0
 115  CONTINUE
C
C     ELEMENTS OF TRIDIAGONAL MATRIX FOR
C     CALCULATING PRESSURE
C
      DO 120 IZ=2,IZM1
      AA(IZ)=-CWC
      BBB(IZ)=1.0
 120  CC(IZ)=-CEC
C
C     RETURN POINT FOR OVERALL ITERATION
C
      ITERA=0
 901  ITERA=ITERA+1
      IF(ITERA.GT.IEND) GO TO 801
      WRITE(6,503) ITERA
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C     BOUNDARY CONDITION FOR P FOR A CAVITY
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 125 IZ=2,IZM1
      PRESSN(IZ,1)=(4.0*PRESSN(IZ,2)-PRESSN(IZ,3))/3.0
     $-(8.0*VNEW(IZ,2)-VNEW(IZ,3))/(3.0*RE*DY)
      PRESSN(IZ,IRMAX)=(4.0*PRESSN(IZ,IRM1)-PRESSN(IZ,IRM2))
     $/3.0+(8.0*VNEW(IZ,IRM1)-VNEW(IZ,IRM2))/(3.0*RE*DY)
 125  CONTINUE
      DO 130 IR=2,IRM1
      PRESSN(1,IR)=(4.0*PRESSN(2,IR)-PRESSN(3,IR))/3.0
     $-(8.0*UNEW(2,IR)-UNEW(3,IR))/(3.0*RE*DX)
      PRESSN(IZMAX,IR)=(4.0*PRESSN(IZM1,IR)-PRESSN(IZM2,IR))
     $/3.0+(8.0*UNEW(IZM1,IR)-UNEW(IZM2,IR))/(3.0*RE*DX)
 130  CONTINUE
      PRESSN(1,1)=(PRESSN(1,2)+PRESSN(2,1))/2.0
      PRESSN(1,IRMAX)=(PRESSN(2,IRMAX)+PRESSN(1,IRM1))/2.0
      PRESSN(IZMAX,IRMAX)=(PRESSN(IZMAX,IRM1)+PRESSN(IZM1,
     $IRMAX))/2.0
```

```
      PRESSN(IZMAX,1)=(PRESSN(IZMAX,2)+PRESSN(IZM1,1))/2.0
C
C     CALCULATION OF NON-HOMOGENEOUS TERM/2.0*(UX*VY-VX*UY)/
C     IN POISSON EQUATION
C
      DO 135 IR=2,IRM1
      DO 135 IZ=2,IZM1
      UXP=(UNEW(IZ+1,IR)-UNEW(IZ-1,IR))/(2.0*DX)
      UYP=(UNEW(IZ,IR+1)-UNEW(IZ,IR-1))/(2.0*DY)
      VXP=(VNEW(IZ+1,IR)-VNEW(IZ-1,IR))/(2.0*DX)
      VYP=(VNEW(IZ,IR+1)-VNEW(IZ,IR-1))/(2.0*DY)
      FNEW(IZ,IR)=2.0*(UXP*VYP-UYP*VXP)
 135  CONTINUE
C
C     RETURN POINT FOR INTERNAL ITERATION OF PRESSURE
C
      ITERB=0
 902  ITERB=ITERB+1
      IF(ITERB.GT.ITERP) GO TO 802
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C     CALCULATION OF PRESSURE FROM POISSON EQUATION
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 140 IR=2,IRM1
      DO 145 IZ=2,M
      D(IZ)=CNE*PRESSN(IZ+1,IR+1)+CSE*PRESSN(IZ+1,IR-1)
     $+CNW*PRESSN(IZ-1,IR+1)+CSW*PRESSN(IZ-1,IR-1)
     $+CNC*PRESSN(IZ,IR+1)+CSC*PRESSN(IZ,IR-1)
      P=FNE*FNEW(IZ+1,IR+1)+FEC*FNEW(IZ+1,IR)
     $+FWC*FNEW(IZ-1,IR)+FNC*FNEW(IZ,IR+1)+FSC*FNEW(IZ,IR-1)
     $+FNW*FNEW(IZ-1,IR+1)+FSE*FNEW(IZ+1,IR-1)+FSW*FNEW(IZ-1,
     $IR-1)+FP*FNEW(IZ,IR)
      P=P*(DX**2)
      D(IZ)=D(IZ)-P
 145  CONTINUE
      D(2)=D(2)+CWC*PRESSN(1,IR)
      D(M)=D(M)+CEC*PRESSN(IZMAX,IR)
      CALL TRIDAG(2,M,AA,BBB,CC,D,T)
      DO 150 IZ=2,IZM1
 150  PRESSN(IZ,IR)=T(IZ)
 140  CONTINUE
      DINITL=0.0
      PDIFF=0.0
      DO 155 IR=2,IRM1
      DO 155 IZ=2,IZM1
      DELP=PRESSN(IZ,IR)-PRESSO(IZ,IR)
      DPMAX=DABS(DELP)
      IF(DPMAX.GT.DINITL) GO TO 803
      GO TO 155
 803  IZPMAX=IZ
      IRPMAX=IR
      PDIFF=DPMAX
```

```
      DINITL=DPMAX
  155 CONTINUE
      IF(PDIFF.LE.0.001) RFP=1.5
      IF(PDIFF.GT.0.001) RFP=1.0
      IF(PDIFF.GT.0.01) RFP=0.5
      IF(PDIFF.GT.0.05) RFP=0.4
      IF(PDIFF.GT.0.1) RFP=0.3
      DO 160 IR=1,IRMAX
      DO 160 IZ=1,IZMAX
      PRESSN(IZ,IR)=PRESSO(IZ,IR)+RFP*
     $(PRESSN(IZ,IR)-PRESSO(IZ,IR))
  160 CONTINUE
      DO 165 IZ=1,IZMAX
      DO 165 IR=1,IRMAX
      PRESSO(IZ,IR)=PRESSN(IZ,IR)
  165 CONTINUE
      IF (PDIFF.LE.0.00001)GO TO 802
      GO TO 901
  802 DO 170 IR=1,IRMAX
      DO 170 IZ=1,IZMAX
  170 PRESSN(IZ,IR)=PRESSN(IZ,IR)-PRESSN(IZMM,IRMAX)
      DO 175 IZ=1,IZMAX
      DO 175 IR=1,IRMAX
  175 PRESSO(IZ,IR)=PRESSN(IZ,IR)
C
C     RETURN POINT FOR INTERNAL ITERATION OF VELOCITY
C
      ITERU=0
  903 ITERU=ITERU+1
      IRMAX1=IRMAX+1
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C     CALCULATION OF VELOCITY FROM MOMENTUM EQUATION
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DO 180 IR=2,IRM1
      DO 180 IZ=2,IZM1
      IZZ=IZ
      IRR=IR
C
C     CALCULATION OF AVERAGE VELOCITIES U&V
C
      UU=(UNEW(IZ+1,IR+1)+4.0*UNEW(IZ+1,IR)+UNEW(IZ+1,IR-1)
     $+4.0*UNEW(IZ,IR+1)+16.0*UNEW(IZ,IR)+4.0*UNEW(IZ,IR-1)
     $+UNEW(IZ-1,IR+1)+4.0*UNEW(IZ-1,IR)+UNEW(IZ-1,IR-1))
     $/36.0
      IF(UU.GT.1.0)UU=1.0
      A=0.5*RE*UU
      VV=(VNEW(IZ+1,IR+1)+4.0*VNEW(IZ+1,IR)+VNEW(IZ+1,IR-1)
     $+4.0*VNEW(IZ,IR+1)+16.0*VNEW(IZ,IR)+4.0*VNEW(IZ,IR-1)
     $+VNEW(IZ-1,IR+1)+4.0*VNEW(IZ-1,IR)+VNEW(IZ-1,IR-1))
     $/36.0
      IF(VV.GT.1.0)VV=1.0
```

```
      B=0.5*RE*VV
      CALL HOMOG
      CALL NHOMOG
      AKK=DABS(A)
      BKK=DABS(B)
      ASQ=AKK**2
      BSQ=BKK**2
      UHNEW=CP1P1*UNEW(IZ+1,IR+1)+CP1PO*UNEW(
     $IZ+1,IR)+CP1M1*UNEW(IZ+1,IR-1)+CPOP1*UNEW(IZ,IR+1)
     $+CPOM1*UNEW(IZ,IR-1)+CM1P1*UNEW(IZ-1,IR+1)+CM1PO
     $*UNEW(IZ-1,IR)+CM1M1*UNEW(IZ-1,IR-1)
      UNH=COEFFP(IZ,IR)*RE*(PRESSN(IZ+1,IR)-
     $PRESSN(IZ-1,IR))/(2.0*DX)
      UNEW1=UHNEW+UNH
      VNEW1=0.5*(VNEW(IZ,IR-1)+VNEW(IZ,IR+1))
     $+0.125*(UNEW(IZ+1,IR+1)-UNEW(IZ-1,IR+1)-UNEW(IZ+1,
     $IR-1)+UNEW(IZ-1,IR-1))
      VHNEW=CP1P1*VNEW(IZ+1,IR+1)+CP1PO*VNEW(
     $IZ+1,IR)+CP1M1*VNEW(IZ+1,IR-1)+CPOP1*VNEW(IZ,IR+1)
     $+CPOM1*VNEW(IZ,IR-1)+CM1P1*VNEW(IZ-1,IR+1)+CM1PO
     $*VNEW(IZ-1,IR)+CM1M1*VNEW(IZ-1,IR-1)
      VNH=COEFFP(IZ,IR)*RE*(PRESSN(IZ,IR+1)-
     $PRESSN(IZ,IR-1))/(2.0*DY)
      VNEW2=VHNEW+VNH
      UNEW2=0.5*(UNEW(IZ+1,IR)+UNEW(IZ-1,IR))
     $+0.125*(VNEW(IZ+1,IR+1)-VNEW(IZ+1,IR-1)-VNEW(IZ-1,
     $IR+1)+VNEW(IZ-1,IR-1))
      UNEW(IZ,IR)=(UNEW1*ASQ+BSQ*UNEW2)/(ASQ+BSQ)
      VNEW(IZ,IR)=(VNEW1*ASQ+BSQ*VNEW2)/(ASQ+BSQ)
 180  CONTINUE
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      DUINTL=0.0
      DVINTL=0.0
      UDIFF=0.0
      VDIFF=0.0
      DO 185 IR=2,IRM1
      DO 185 IZ=2,IZM1
      DELU=UNEW(IZ,IR)-UOLD(IZ,IR)
      DELV=VNEW(IZ,IR)-VOLD(IZ,IR)
      DUMAX=DABS(DELU)
      DVMAX=DABS(DELV)
      IF(DUMAX.GT.DUINTL) GO TO 804
      GO TO 805
 804  IZUMAX=IZ
      IRUMAX=IR
      UDIFF=DUMAX
      DUINTL=DUMAX
 805  IF(DVMAX.GT.DVINTL) GO TO 806
      GO TO 185
 806  IZVMAX=IZ
      IRVMAX=IR
```

```
      VDIFF=DVMAX
      DVINTL=DVMAX
  185 CONTINUE
      IF(UDIFF.LE.0.001) RFU=1.0
      IF(UDIFF.GT.0.001)RFU=0.6
      IF(UDIFF.GT.0.01)RFU=0.4
      IF(UDIFF.GT.0.1) RFU=0.3
      IF(VDIFF.LE.0.001) RFV=1.0
      IF(VDIFF.GT.0.001)RFV=0.6
      IF(VDIFF.GT.0.01)RFV=0.4
      IF(VDIFF.GT.0.1)RFV=0.3
      DO 190 IR=2,IRM1
      DO 190 IZ=2,IZM1
      UNEW(IZ,IR)=UOLD(IZ,IR)+RFU*(UNEW(IZ,IR)-UOLD(IZ,IR))
      VNEW(IZ,IR)=VOLD(IZ,IR)+RFV*(VNEW(IZ,IR)-VOLD(IZ,IR))
  190 CONTINUE
      DO 195 IR=1,IRMAX
      DO 195 IZ=1,IZMAX
      UOLD(IZ,IR)=UNEW(IZ,IR)
      VOLD(IZ,IR)=VNEW(IZ,IR)
  195 CONTINUE
      IF(ITERU.GT.ITRV) GO TO 807
      IF(UDIFF.LE.TOLUV) GO TO 808
      GO TO 903
  808 IF(VDIFF.LE.TOLUV) GO TO 807
      GO TO 903
C
C     CALCULATION OF VORTICITY AND STREAM-FUNCTION
C
  807 DO 200 IR=2,IRM1
      DO 200 IZ=2,IZM1
      ZETA(IZ,IR)=(UNEW(IZ,IR+1)-UNEW(IZ,IR-1))/(2.*DY)
     $-(VNEW(IZ+1,IR)-VNEW(IZ-1,IR))/(2.*DX)
  200 CONTINUE
      DO 205 IZ=2,IZM1
      ZETA(IZ,1)=-2.*(DY-PSIN(IZ,2))/DYY
      ZETA(IZ,IRMAX)=2.*PSIN(IZ,IRM1)/DYY
  205 CONTINUE
      DO 210 IR=2,IRM1
      ZETA(1,IR)=2.*PSIN(2,IR)/DXX
      ZETA(IZMAX,IR)=2.*PSIN(IZM1,IR)/DXX
  210 CONTINUE
      ZETA(1,1)=-2.0/DX
      ZETA(IZMAX,1)=-2.0/DX
      ITERPZ=0
  904 ITERPZ=ITERPZ+1
      DO 215 IR=2,IRM1
      DO 220 IZ=2,IZM1
      PS(IZ)=CNE*PSIN(IZ+1,IR+1)+CSE*PSIN(IZ+1,IR-1)
     $+CNW*PSIN(IZ-1,IR+1)+CSW*PSIN(IZ-1,IR-1)
     $+CNC*PSIN(IZ,IR+1)+CSC*PSIN(IZ,IR-1)
```

```
     $-(FNE*ZETA(IZ+1,IR+1)+FEC*ZETA(IZ+1,IR)
     $+FWC*ZETA(IZ-1,IR)+FNC*ZETA(IZ,IR+1)+FSC*ZETA
     $(IZ,IR-1)+FNW*ZETA(IZ-1,IR+1)+FSE*ZETA(IZ+1,IR-1)+
     $FSW*ZETA(IZ-1,IR-1)+FP*ZETA(IZ,IR))*DXX
 220  CONTINUE
      PS(2)=PS(2)+CWC*PSIN(1,IR)
      PS(M)=PS(M)+CEC*PSIN(IZMAX,IR)
      CALL TRIDAG(2,M,AA,BBB,CC,PS,T)
      DO 225 IZ=2,IZM1
 225  PSIN(IZ,IR)=T(IZ)
 215  CONTINUE
      IF(ITERPZ.GT.30)GO TO 901
      GO TO 904
C
C     FORCE BALANCE CHECK ON CAVITY
C
 801  SUMP1=0.
      SUMP2=0.
      DO 230 IZ=1,IZMAX
      SUMP1=SUMP1+PRESSN(IZ,1)
      SUMP2=SUMP2+PRESSN(IZ,IRMAX)
 230  CONTINUE
      SUMP1=SUMP1*DX
      SUMP2=SUMP2*DX
      DIFFP=SUMP1-SUMP2
      SUMU1=0.
      SUMU2=0.
      DO 235 IR=1,IRMAX
      SUMU1=SUMU1-VNEW(IZM1,IR)
      SUMU2=SUMU2+VNEW(2,IR)
 235  CONTINUE
      SUMU1=SUMU1*DY/(DX*RE)
      SUMU2=SUMU2*DY/(DX*RE)
      DIFFU=SUMU1-SUMU2
      CHECKY=DABS(DIFFP)+DABS(DIFFU)
      SUMP3=0.0
      SUMP4=0.0
      DO 240 IR=1,IRMAX
      SUMP3=SUMP3+PRESSN(1,IR)
      SUMP4=SUMP4+PRESSN(IZMAX,IR)
 240  CONTINUE
      SUMP3=SUMP3*DY
      SUMP4=SUMP4*DY
      SUMV3=0.0
      SUMV4=0.0
      DO 245 IZ=1,IZMAX
      SUMV3=SUMV3-UNEW(IZ,IRM1)
      SUMV4=SUMV4+UNEW(IZ,2)-UNEW(IZ,1)
 245  CONTINUE
      SUMV3=SUMV3*DX/(DY*RE)
      SUMV4=SUMV4*DX/(DY*RE)
```

```
      CHECKX=DABS(SUMP3)-DABS(SUMP4)-DABS(SUMV3)+DABS(SUMV4)
      WRITE(6,504)
      DO 250 IZ=1,IZMAX
  250 WRITE(6,505) (PRESSN(IZ,IR),IR=1,IRMAX)
      WRITE(6,506)
      DO 255 IZ=1,IZMAX
  255 WRITE(6,505) (UNEW(IZ,IR),IR=1,IRMAX)
      WRITE(6,507)
      DO 260 IZ=1,IZMAX
  260 WRITE(6,505) (VNEW(IZ,IR),IR=1,IRMAX)
      WRITE(6,508)
      DO 265 IZ=1,IZMAX
  265 WRITE(6,505)(PSIN(IZ,IR),IR=1,IRMAX)
      WRITE(6,509)
      DO 270 IZ=1,IZMAX
  270 WRITE(6,505)(ZETA(IZ,IR),IR=1,IRMAX)
      WRITE(6,510) CHECKX
      WRITE(6,511) CHECKY
C
C     FORMATS
C
  500 FORMAT(1X,8F9.6)
  501 FORMAT('IZMAX=',I3,/'IRMAX=',I3/'ITERP=',I3/
     $'IEND =',I3/'ITRV =',I3/'X     =',F9.6/'Y     =',F9.6/
     $'UP=',F9.4/'RE=',F9.4/'TOLUV=',F6.4/)
  502 FORMAT('DX=',F6.4/'DY=',F6.4/)
  503 FORMAT(//'ITERA=',I3)
  504 FORMAT(/'PRESSURE IS')
  505 FORMAT(/'IZ=',I3,3X,11F9.5,9(/9X,10F9.5,))
  506 FORMAT(/'VELOCITY U IS')
  507 FORMAT(/'VELOCITY V IS')
  508 FORMAT(/'STREAM-FUNCTION IS')
  509 FORMAT(/'VORTICITY IS')
  510 FORMAT('NET FORCE IN X-DIRECTION IS',E14.7)
  511 FORMAT('NET FORCE IN Y-DIRECTION IS',E14.7)
      CALL EXIT
      END
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C
C     SUBROUTINE TO CALCULATE PRESSURE USING IMPLICIT
C     METHOD.
C
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE TRIDAG(IF,L,AAA,BBB,C,D,V)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION AAA(61),BBB(61),C(61),D(61),V(61),BETA(61),
     $GAMMA(61)
      BETA(IF)=BBB(IF)
      GAMMA(IF)=D(IF)/BETA(IF)
```

```
      IFP1=IF+1
      DO 1 I=IFP1,L
      BETA(I)=BBB(I)-AAA(I)*C(I-1)/BETA(I-1)
 1    GAMMA(I)=(D(I)-AAA(I)*GAMMA(I-1))/BETA(I)
      V(L)=GAMMA(L)
      LAST=L-IF
      DO 2 K=1,LAST
      I=L-K
 2    V(I)=GAMMA(I)-C(I)*V(I+1)/BETA(I)
      RETURN
      END
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C
C     SUBROUTINE TO CALCULATE THE COEFFICIENTS FOR
C     THE MOMENTUM EQUATION.
C     THE COEFFICIENTS ARE STORED IN A DATA-BANK FOR
C     VARIOUS VALUES OF A AND B. DEPENDING ON THE
C     VALUE OF A AND B IN EACH ELEMENT THE COEFFICIENTS
C     ARE INTERPOLATED FROM THE DATA-BANK.
C
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE HOMOG
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      COMMON/COMD/DCM1P1(74,74),DCM1P0(74,74),DCM1M1(74,74),
     $DCP0P1(74,74),DCP0M1(74,74),DCP1P1(74,74),DCP1P0(74,74),
     $DCP1M1(74,74)
      COMMON/COMF/A,B,IZZ,IRR,AKK,BKK
     $,ITER,IZMAX,IRMAX,DX,DY
      COMMON/COMC/ CP1P1,CP1P0,CP1M1,CM1P1
     $,CM1P0,CM1M1,CP0P1,CP0M1
CSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS
C                       FIND THE LOCATION OF
C                       CORRESPONDANCE POINT IN DATA BANK
CSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS
      AH=A*DX
      BK=B*DY
      AAH=DABS(AH)
      BBK=DABS(BK)
   16 AAC=AAH/0.01
      IF(AAH.LT.0.01)GO TO 10
      IF(AAH.LE.0.1.AND.AAH.GE.0.01)GO TO 11
      IF(AAH.LE.1.0.AND.AAH.GT.0.1)GO TO 12
      IF(AAH.LE.10.0.AND.AAH.GT.1.0)GO TO 13
      IF(AAH.LT.100.0.AND.AAH.GT.10.0)GO TO 14
      IF(AAH.GE.100.0)AAH=99.9999
      GO TO 16
   10 ADX=1.0
      NOA=1
      AAX=AAC
```

```
      GO TO 15
   11 ADX=0.5
      PP=(AAC-1.0)/0.5
      NP=PP
      AAX=(PP-NP)*0.5
      NOA=NP+2
      GO TO 15
   12 ADX=5.0
      PP=(AAC-10.0)/5.0
      NP=PP
      AAX=(PP-NP)*5.0
      NOA=NP+20
      GO TO 15
   13 ADX=50.0
      PP=(AAC-100.0)/50.0
      NP=PP
      AAX=(PP-NP)*50.0
      NOA=NP+38
      GO TO 15
   14 ADX=500.0
      PP=(AAC-1000.0)/500.0
      NP=PP
      AAX=(PP-NP)*500.0
      NOA=NP+56
   15 CONTINUE
   26 BBC=BBK/0.01
      IF(BBK.LT.0.01) GO TO 20
      IF(BBK.LE.0.1.AND.BBK.GE.0.01)GO TO 21
      IF(BBK.LE.1.0.AND.BBK.GT.0.1)GO TO 22
      IF(BBK.LE.10.0.AND.BBK.GT.1.0)GO TO 23
      IF(BBK.LT.100.0.AND.BBK.GT.10.0)GO TO 24
      IF(BBK.GE.100.0)BBK=99.9999
      GO TO 26
   20 BDY=1.0
      NOB=1
      BBY=BBC
      GO TO 25
   21 BDY=0.5
      PP=(BBC-1.0)/0.5
      NP=PP
      BBY=(PP-NP)*0.5
      NOB=NP+2
      GO TO 25
   22 BDY=5.0
      PP=(BBC-10.0)/5.0
      NP=PP
      BBY=(PP-NP)*5.0
      NOB=NP+20
      GO TO 25
   23 BDY=50.0
      PP=(BBC-100.0)/50.0
```

```
      NP=PP
      BBY=(PP-NP)*50.0
      NOB=NP+38
      GO TO 25
   24 BDY=500.0
      PP=(BBC-1000.0)/500.0
      NP=PP
      BBY=(PP-NP)*500.0
      NOB=NP+56
   25 CONTINUE
CSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS
C                     INTERPOLATION OF THE LOCATED POINT
C                     IN DATA BANK
CSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS
      NOA1=NOA+1
      NOB1=NOB+1
      XODX=AAX/ADX
      YODY=BBY/BDY
      C00=1-XODX-YODY+(XODX*YODY)
      C10=XODX-(XODX*YODY)
      C11=XODX*YODY
      C01=YODY-(XODX*YODY)
      CM1P1=DCM1P1(NOA,NOB)*C00+DCM1P1(NOA1,NOB)*C10+
     1          DCM1P1(NOA1,NOB1)*C11+DCM1P1(NOA,NOB1)*C01
      CM1P0=DCM1P0(NOA,NOB)*C00+DCM1P0(NOA1,NOB)*C10+
     2          DCM1P0(NOA1,NOB1)*C11+DCM1P0(NOA,NOB1)*C01
      CM1M1=DCM1M1(NOA.NOB)*C00+DCM1M1(NOA1,NOB)*C10+
     3          DCM1M1(NOA1,NOB1)*C11+DCM1M1(NOA,NOB1)*C01
      CP0P1=DCP0P1(NOA,NOB)*C00+DCP0P1(NOA1,NOB)*C10+
     4          DCP0P1(NOA1,NOB1)*C11+DCP0P1(NOA,NOB1)*C01
      CP0M1=DCP0M1(NOA,NOB)*C00+DCP0M1(NOA1,NOB)*C10+
     5          DCP0M1(NOA1,NOB1)*C11+DCP0M1(NOA,NOB1)*C01
      CP1P1=DCP1P1(NOA,NOB)*C00+DCP1P1(NOA1,NOB)*C10+
     6          DCP1P1(NOA1,NOB1)*C11+DCP1P1(NOA,NOB1)*C01
      CP1P0=DCP1P0(NOA,NOB)*C00+DCP1P0(NOA1,NOB)*C10+
     7          DCP1P0(NOA1,NOB1)*C11+DCP1P0(NOA,NOB1)*C01
      CP1M1=DCP1M1(NOA,NOB)*C00+DCP1M1(NOA1,NOB)*C10+
     8          DCP1M1(NOA1,NOB1)*C11+DCP1M1(NOA,NOB1)*C01
CSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS
C                 TURN TO THE EXACT DIRECTION OF
C                      COEFFICIENTS----A AND B
CSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS
      IF(A.GE.0.0)GO TO 33
      CNE=CP1P1
      CP1P1=CM1P1
      CM1P1=CNE
      CEC=CP1P0
      CP1P0=CM1P0
      CM1P0=CEC
      CSE=CP1M1
      CP1M1=CM1M1
```

C-5

```
      CM1M1=CSE
   33 CONTINUE
      IF(B.GE.0.0)GO TO 34
      CNW=CM1P1
      CM1P1=CM1M1
      CM1M1=CNW
      CNC=CP0P1
      CP0P1=CP0M1
      CP0M1=CNC
      CNE=CP1P1
      CP1P1=CP1M1
      CP1M1=CNE
   34 CONTINUE
      RETURN
      END
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C
C     SUBROUTINE TO CALCULATE THE COEFFICIENT OF
C     THE INHOMOGENEOUS TERM IN THE MOMENTUM
C     EQUATION
C
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
C+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      SUBROUTINE NHOMOG
      IMPLICIT DOUBLE PRECISION (A - H, O - Z)
      COMMON/COMF/A,B,IZZ,IRR,AKK,BKK
     $,ITER,IZMAX,IRMAX,DX,DY
      COMMON/COME/COEFFP(61,61)
      SUM=0.
      DO 10 L=1,35,2
      AL=FLOAT(L)
      H=AL*1.5707963/DX
      QLS=(A**2)+(B**2)+(H**2)
      QL=DSQRT(QLS)
      X1=DEXP((QL-B)*DY)-DEXP(-(QL-B)*DY)
      X2=DEXP((QL+B)*DY)-DEXP(-(QL+B)*DY)
      X3=DEXP(2.0*QL*DY)-DEXP(-2.0*QL*DY)
      X4=(X1+X2)/X3
      X5=X4-1.0
      X6=DEXP(A*DX)+DEXP(-A*DX)
      X7=H*X6/DX
      X8=(QLS-B**2)
      XX8=H**2+A**2
      X9=X7*X5/(X8*XX8)
      X0=X9*DSIN(AL*1.5707963)
      SUM=SUM+X0
 10   CONTINUE
      COEFFP(IZZ,IRR)=SUM
      RETURN
      END
```

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